Math 2153, Final Exam, Fall 2012 Name KEY

Instructions: Show all work. Answer all parts of each question. Use exact values unless specifically asked to round or do the problem numerically. Some problems will ask you to complete the question in its entirety and some to merely "set up". Read the instructions on the problem very carefully.

1. For the scalar function $f(x,y,z) = xyz + x^2 + z\sin(y)$ and the vector field $\vec{G}(x,y,z) = xy\vec{i} + e^z\vec{j} - 2xz^3\vec{k}$, calculate the following:
   a. $\vec{\nabla}f$ (5 points)

   $(yz+2x)\hat{i} + (xz + z\cos y)\hat{j} + (xy + \sin y)\hat{k}$

   b. $\vec{\nabla} \times \vec{G}$ (8 points)

   $$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & e^z & -2xz^3 \end{vmatrix} = (0-e^z)\hat{i} - (-2z^3-0)\hat{j} + (0-x)\hat{k}$$

   $$\boxed{-e^z\hat{i} + 2z^3\hat{j} - x\hat{k}}$$

   c. $\vec{\nabla} \cdot \vec{G}$ (5 points)

   $$y + 0 - 6xz^2 = \boxed{y - 6xz^2}$$

   d. $(\vec{\nabla}f) \times (\vec{\nabla} \times \vec{G})$ (7 points)

   $$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ yz+2x & xz+z\cos y & xy+\sin y \\ -e^z & 2z^3 & -x \end{vmatrix} =$$

   $$\boxed{\begin{aligned}&(-x^2z - xz\cos y - 2xyz^3 - 2z^3\sin y)\hat{i} - (-xyz - 2x^2 + xye^z + e^z\sin y)\hat{j} + \\ &(2yz^4 + 4xz^3 + xze^z + ze^z\cos y)\hat{k}\end{aligned}}$$

2. Determine if the vector field $\vec{F}(x,y,z) = \frac{1}{y}\vec{\imath} - \frac{x}{y^2}\vec{\jmath} + (2z-1)\vec{k}$ is conservative. If so, find the potential function. (10 points)

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1/y & -x/y^2 & 2z-1 \end{vmatrix} = (0-0)\hat{\imath} - (0-0)\hat{\jmath} + \left(-1/y^2 - (-1/y^2)\right)\hat{k}$$

$$= \vec{0} \quad \text{conservative}$$

$$\int \frac{1}{y}\,dx = \frac{x}{y} + C(y,z)$$

$$\int -\frac{x}{y^2}\,dy = \frac{x}{y} + C(x,z)$$

$$\int 2z-1\,dz = z^2 - z + C(x,y)$$

$$\boxed{f(x,y,z) = \frac{x}{y} + z^2 - z + K}$$

3. Using the line integral $\int_C \rho(x,y,z)ds$ calculate the mass of the wire for $\rho(x,y,z) = k + z$ and along the path $C: \vec{r}(t) = 2\cos(t)\,\vec{\imath} + 2\sin(t)\,\vec{\jmath} + 3t\vec{k}, k > 0, 0 \le t \le 4\pi$. Reduce line integral to a definite integral in a single variable and stop; you do not need to integrate. (20 points)

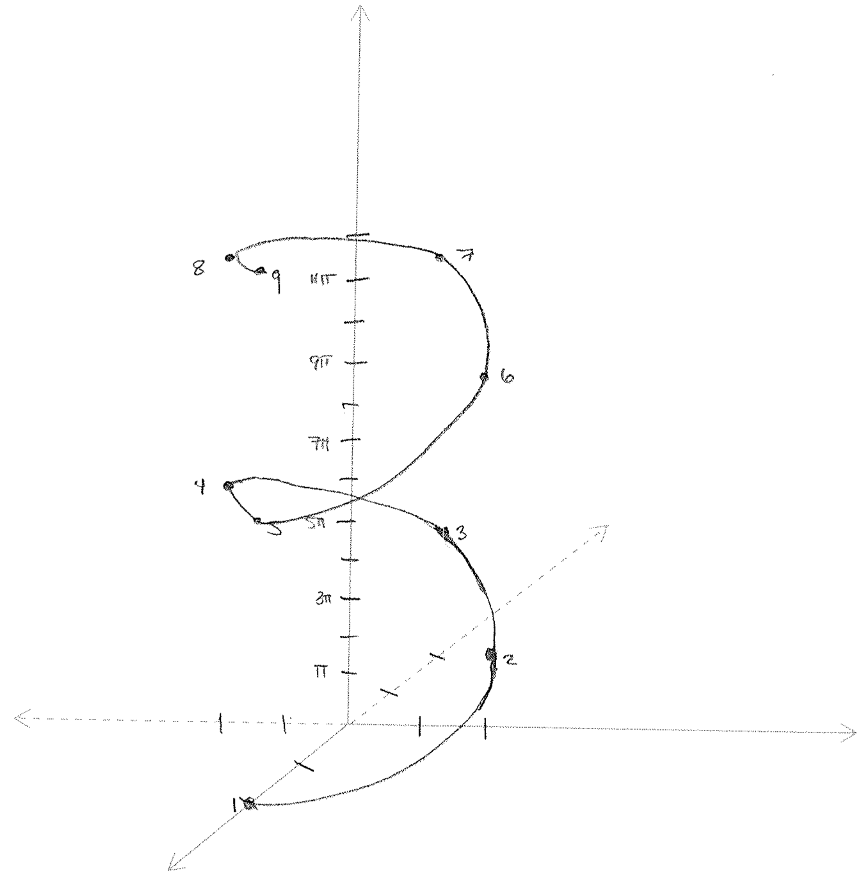
$$r'(t) = -2\sin t\,\hat{\imath} + 2\cos t\,\hat{\jmath} + 3\hat{k} \qquad ds = \|r'(t)\|\,dt$$

$$\|r'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 9} = \sqrt{4+9} = \sqrt{13}$$

$$\int_C \rho\,ds = \boxed{\int_0^{4\pi} (K+3t)\sqrt{13}\,dt}$$

4. Sketch the path of the helix in Problem #3, by plotting the length of the wire measured in the line integral (2 turns of the helix). You should plot at least 9 points. You will want to scale your z-axis differently than your x- and y-axes. (10 points)

t	x	y	z
0	2	0	0
$\pi/2$	0	2	$3\pi/2$
π	-2	0	3π
$3\pi/2$	0	-2	$9\pi/2$
2π	2	0	6π
$5\pi/2$	0	2	$15\pi/2$
3π	-2	0	9π
$7\pi/2$	0	-2	$21\pi/2$
4π	2	0	12π



5. Calculate the value of the line integral $\int_C \vec{F}\cdot d\vec{r}$ for $\vec{F}(x,y) = \frac{2x}{(x^2+y^2)^2}\vec{i} + \frac{2y}{(x^2+y^2)^2}\vec{j}$ along the path C: *circle* $(x-4)^2 + (y-5)^2 = 9$ clockwise from (7,5) to (1,5). [Hint: is the field conservative?] (20 points)

$$\frac{\partial}{\partial x}\left[\frac{2y}{(x^2+y^2)^2}\right] = \frac{\partial}{\partial x}\left[2y(x^2+y^2)^{-2}\right] = 2y(-2)(x^2+y^2)^{-3}(2x) = \frac{-8xy}{(x^2+y^2)^3} \checkmark$$

$$\frac{\partial}{\partial y}\left[\frac{2x}{(x^2+y^2)^2}\right] = \frac{\partial}{\partial y}\left[2x(x^2+y^2)^{-2}\right] = 2x(-2)(x^2+y^2)^{-3}(2y) = \frac{-8xy}{(x^2+y^2)^3} \checkmark$$

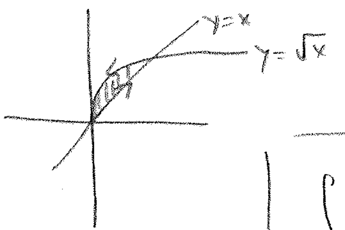
Conservative

$$\int \frac{2x}{(x^2+y^2)^2}\,dx \qquad u = x^2+y^2,\; du = 2x,\; \int \frac{1}{u^2}\,du = -\frac{1}{u} + C \qquad = -\frac{1}{x^2+y^2} + C(y)$$

$$\int \frac{2y}{(x^2+y^2)^2}\,dy \qquad u = x^2+y^2,\; du = 2y,\; \int \frac{1}{u^2}\,du = -\frac{1}{u} + C \qquad = -\frac{1}{x^2+y^2} + C(x)$$

$$f(x,y) = \frac{-1}{x^2+y^2} \qquad f(7,5) - f(1,5) = \frac{-1}{49+25} - \left(-\frac{1}{1^2+25}\right) = \frac{1}{26} - \frac{1}{74} = \boxed{\frac{12}{481}}$$

6. Calculate the value of the line integral $\int_C \sin(x)\cos(y)\,dx + (xy + \cos(x)\sin(y))dy$ on the path C: *boundary of the region lying between* $y = x$ *and* $y = \sqrt{x}$. Reduce the line integral to a single double integral or a pair of single variable definite integrals in one variable and stop; you do not need to integrate. (20 points)



$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = y - \cancel{\sin x \sin y} - (\cancel{-\sin x \sin y}) = y$$

$$\boxed{\int_0^1 \int_x^{\sqrt{x}} y \, dy \, dx}$$

7. Consider the surface described parametrically by $\vec{r}(u,v) = \sin(u)\cos(v)\,\vec{\imath} + u\vec{\jmath} + \sin(u)\sin(v)\,\vec{k}$, $0 \le u \le \pi, 0 \le v \le 2\pi$. Find the following:

   a. $\vec{N}$ to the surface, and evaluate it at the point $\left(\frac{1}{2}, \frac{\pi}{4}, \frac{1}{2}\right)$. (10 points)

$\vec{r_u} = \cos u \cos v\,\hat{i} + 1\hat{j} + \cos u \sin v\,\hat{k}$

$\vec{r_v} = -\sin u \sin v + 0\hat{j} + \sin u \cos v\,\hat{k}$

$$\vec{r_u} \times \vec{r_v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos u \cos v & 1 & \cos u \sin v \\ -\sin u \sin v & 0 & \sin u \cos v \end{vmatrix} =$$

$\frac{1}{2} = \sin u \cos v \Rightarrow \frac{1}{\sqrt{2}} \cos v$

$u = \pi/4$ $\qquad \frac{1}{\sqrt{2}} = \cos v$

$\frac{1}{2} = \sin u \sin v$ $\qquad v = \pi/4$

$(u, v) = (\pi/4, \pi/4)$

$$(\sin u \cos v - 0)\hat{i} - (\cos u \cos^2 v \sin u + \cos u \sin^2 v \sin u)\hat{j} + (0 + \sin u \sin v)\hat{k}$$

$\cos u \sin u$

$\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}$

b. The equation of the tangent line at the given point, and the equation of the normal line at the same point. (5 points)

$$\frac{1}{2}\left(x-\frac{1}{2}\right) - \frac{1}{2}\left(y-\frac{\pi}{4}\right) + \frac{1}{2}\left(z-\frac{1}{2}\right) = 0 \quad \text{tangent line}$$

$$\vec{r}(t) = \left(\tfrac{1}{2}t+\tfrac{1}{2}\right)\hat{\imath} + \left(\tfrac{\pi}{4}-\tfrac{1}{2}t\right)\hat{\jmath} + \left(\tfrac{1}{2}t+\tfrac{1}{2}\right)\hat{k} \quad \text{parametric}$$

or

$$\frac{x-\frac{1}{2}}{\frac{1}{2}} = \frac{y-\frac{\pi}{4}}{-\frac{1}{2}} = \frac{z-\frac{1}{2}}{\frac{1}{2}} \quad \text{symmetric}$$

normal line

c. Find the area of the surface with the given limits in u and v. Set up the integral, reduced as much as possible algebraically, and stop; you do not need to integrate. (10 points)

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{\sin^2 u \cos^2 v + \cos^2 u \sin^2 u + \sin^2 u \sin^2 v} =$$

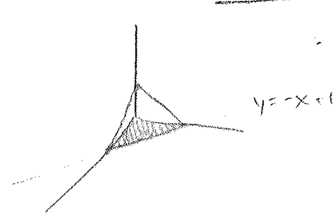
$$\sqrt{\sin^2 u\left(\cos^2 v + \cos^2 u + \sin^2 v\right)} = \sin u\sqrt{1+\cos^2 v}$$

$$\boxed{\int_0^{2\pi}\int_0^{\pi} \sin u\sqrt{1+\cos^2 v}\, du\, dv}$$

8. Evaluate the flux integral $\iint_S \vec{F}\cdot\vec{N}\,dS$, for the field $\vec{F}(x,y,z) = 3z\vec{\imath} - 4\vec{\jmath} + y\vec{k}$ and for the surface $S: x+y+z=1$, in the first octant. (20 points)

$$\vec{N}\,dS = \vec{\nabla}G(x,y,g(x,y))\,dA \qquad z = 1-x-y \qquad G(x,y,z) = x+y+z-1 \qquad \boxed{-\frac{4}{3}}$$

$$\nabla G = 1\hat{\imath} + 1\hat{\jmath} + \hat{k} \qquad x+y=1 \qquad y=-x+1$$



$$\int_0^1\int_0^{-x+1} \langle 3(1-x-y), -4, y\rangle\langle 1,1,1\rangle\, dy\,dx =$$

$$\int_0^1\int_0^{-x+1} 3-3x-3y-4+y\, dy\,dx = \int_0^1\int_0^{-x+1} -1-3x-2y\, dy\,dx =$$

$$\int_0^1 -1(-x+1) - 3x(-x+1) - (-x+1)^2\, dx = \int_0^1 x-1+3x^2-3x-x^2+2x-1\, dx = \int_0^1 2x^2-2\, dx =$$

#8 $\frac{2}{3}x^3 - 2x\Big|_0^1 = \frac{2}{3} - 2 = \boxed{-\frac{4}{3}}$

9. Recalculate the flux integral in problem #8 with the same field through the closed surface bounded by the plane and the coordinate planes using the Divergence Theorem. If your answer for #8 and #9 disagree, explain why. (12 points)

$$\iint_S \vec{F} \cdot \vec{N}\, dS = \iiint_Q \operatorname{div} \vec{F}\, dV$$

$$\vec{\nabla} \cdot \vec{F} = 0 + 0 + 0$$

$$\int_0^1 \int_0^{-x+1} \int_0^{1-x-y} 0 \; dz\, dy\, dx = \boxed{0}$$

These answers disagree because in #8 we are calculating flux through the single surface $x+y+z=1$ and not also the coordinate planes, which this one does include.

10. Use Stokes' Theorem to calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$ for the field $\vec{F}(x,y,z) = \arctan\left(\frac{x}{y}\right)\vec{\imath} + ln\sqrt{x^2+y^2}\vec{\jmath} + k$ along the path $C: triangle\ with\ vertices\ (0,0,0), (0,2,0), (1,1,1)$. Set up the double integral and stop; you do not need to integrate. [Hint: find two vectors in the plane and use a cross product to find the normal vector to the plane. You do not need to find the equation of the plane, but do consider the orientation of the normal.] (20 points)

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \arctan(\frac{y}{x}) & \frac{1}{2}\ln(x^2+y^2) & 1 \end{vmatrix} = (0-0)\hat{\imath} - (0-0)\hat{\jmath} + \left(\frac{1}{\cancel{2}}\frac{1}{x^2+y^2}\cancel{2}x - \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}\frac{x}{x}\right)\hat{k}$$

$$0\hat{\imath} + 0\hat{\jmath} + \left(\frac{x}{x^2+y^2} - \frac{x}{x^2+y^2}\right)\hat{k} = \vec{0}$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{N}\, dS$$

$$\langle 0,2,0 \rangle \times \langle 1,1,1 \rangle$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 2\hat{\imath} - 0\hat{\jmath} - 2\hat{k}$$

$2(x-0) - 2(z-0) = 0$

$2x = 2z$

$x = z$

$G(x,y,z) = z - x$

$$\nabla G = -1\hat{\imath} + 0\hat{\jmath} + 1\hat{k}$$

$$\int_0^1 \int_0^2 \langle 0,0,0 \rangle \cdot \langle -1,0,-1 \rangle \, dy\, dx$$

$$= \boxed{\int_0^1 \int_0^2 0 \; dy\, dx = 0}$$

11. Find an equation for the curvature of the cycloid $\vec{r}(t) = (t - \sin(t))\vec{\imath} + (1 - \cos(t))\vec{\jmath}$. (8 points)

$r'(t) = (1-\cos t)\hat{\imath} + \sin t\,\hat{\jmath} + 0\hat{k}$ $\quad \|r'(t)\| = \sqrt{(1-\cos t)^2 + \sin^2 t} =$

$r''(t) = \sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + 0\hat{k}$ $\quad \sqrt{1-2\cos t + \cos^2 t + \sin^2 t} =$

$\sqrt{2-2\cos t} = (\sqrt{2})\sqrt{1-\cos t}$

$$r' \times r'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1-\cos t & \sin t & 0 \\ \sin t & \cos t & 0 \end{vmatrix} = (0-0)\hat{\imath} - (0-0)\hat{\jmath} + (\cos t - \cos^2 t - \sin^2 t)\hat{k} =$$

$+ \cos t - 1$

$$K = \frac{\cancel{|\cos t - 1|}}{2\sqrt{2}\,(1-\cancel{\cos t})\sqrt{1-\cos t}} = \boxed{\frac{\sqrt{2}}{4\sqrt{1-\cos t}}}$$

12. Determine if $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+4y^2}$ exists or does not exist. Test *at least* two paths. (10 points)

$x=0 \quad \lim_{y\to 0} \frac{0}{4y^2} = 0$

$y=x \quad \lim_{x\to 0} \frac{x^3}{x^4+4x^2} = \frac{x^3}{x^2(x^2+4)} = \frac{x}{x^2+4} = 0$

$y=0 \quad \lim_{x\to 0} \frac{0}{x^4} = 0$

$y=kx^2 \quad \lim_{x\to 0} \frac{kx^4}{x^4+4k^2x^4} = \frac{k\cancel{x^4}}{\cancel{x^4}(1+4k^2)} =$

$\frac{k}{1+4k^2} \neq 0$ unless $k=0$

$\boxed{\text{DNE}}$

13. Find the partial derivative f_{xyyz} for the function $f(x, y, z) = e^{-x}\sin(yz)$. (12 points)

$f_x = -e^{-x}\sin(yz)$

$f_{xy} = -ze^{-x}\cos(yz)$

$f_{xyy} = +z^2e^{-x}\sin(yz)$

$f_{xyyz} = \boxed{2ze^{-z}\sin yz + z^2e^{-x}\cos(yz)\cdot y}$

14. Find the normal vector to the graph $z = x^2y^4$ at the point (1,2,16). Is this vector oriented to the outward surface or the inward surface? Then give the equation of the tangent plane. (15 points)

$F(x,y,z) = x^2y^4 - z$

$\nabla F = 2xy^4\hat{i} + 4x^2y^3\hat{j} - 1\hat{k}$

$\nabla F(1,2,16) = 32\hat{i} + 32\hat{j} - 1\hat{k}$

downward oriented

$$\boxed{32(x-1) + 32(y-2) - 1(z-16) = 0}$$

15. Find an critical points for the graph $f(x,y) = 120x + 120y - xy - x^2 - y^2$. Characterize the critical points as a relative maximum, relative minimum, a saddle point, or cannot be determined. (12 points)

$f_x = 120 - y - 2x = 0$

$f_y = 120 - x - 2y = 0$

$2x + y = 120$

$(x-2)\quad x + 2y = 120$

$2x + y = 120$

$-2x - 4y = -240$

$-3y = -120$

$y = 40$

$2x + 40 = 120$

$2x = 80$

$x = 40$

$$\boxed{(40,40) \text{ relative maximum}}$$

$f_{xx} = -2$

$f_{xy} = -1$

$f_{yy} = -2$

$f_{xx} \cdot f_{yy} - (f_{xy})^2 =$

$(-2)(-2) - (-1)^2 = 4 - 1 = 3 > 0$ not a saddle

$f_{xx} < 0$ max

16. Integrate. (7 points each)

a. $\int_0^4 \int_1^2 x^2 - 2y^2 + 1\, dxdy$

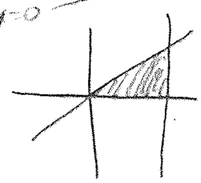
$$\frac{1}{3}x^3 - 2y^2x + x \Big|_1^2 = \frac{8}{3} - 4y^2 + 2 - \frac{1}{3} + 2y^2 - 1 =$$

$$\int_0^4 \frac{10}{3} - 2y^2\, dy = \frac{10}{3}y - \frac{2}{3}y^3 \Big|_0^4 = \frac{40}{3} - \frac{128}{3} = -\frac{88}{3} = \boxed{-\frac{88}{3}}$$

intersect → $x=1$

b. $\int_0^1 \int_y^1 \sin(x^2)\, dxdy$ [Hint: you will want to change the order of integration.]

$y=0$ → $x=y$



$$\int_0^1 \int_0^x \sin(x^2)\, dy\, dx =$$

$$\int_0^1 x \sin(x^2)\, dx \qquad \begin{array}{l} u = x^2 \\ du = 2x\,dx \\ x\,dx = \frac{1}{2}du \end{array} \qquad \int \frac{1}{2}\sin u\, du$$

$\boxed{\text{X in radians!}}$

$$-\frac{1}{2}\cos(x^2)\Big|_0^1 = -\frac{1}{2}\cos(1) + \frac{1}{2}(1) =$$

$$\boxed{\frac{1}{2} - \frac{1}{2}\cos(1)}$$

c. $\int_0^4 \int_0^{\pi/2} \int_0^{1-x} x\cos(y)\, dzdydx$

$$\int_0^4 \int_0^{\pi/2} xz\cos y \Big|_0^{1-x} dy\, dx = \int_0^4 \int_0^{\pi/2} (x - x^2)\cos y\, dy\, dx$$

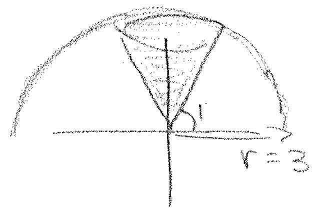
$$\int_0^4 \sin y \Big|_0^{\pi/2} (x - x^2)\, dx = \int_0^4 (x - x^2)\, dx = \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_0^4$$

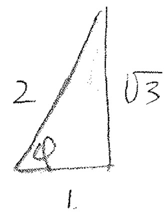
$$= 8 - \frac{64}{3} = \frac{24 - 64}{3} = \boxed{-\frac{40}{3}}$$

17. Find the volume of the region bounded by the cone $z = \sqrt{\frac{x^2+y^2}{3}}$, and the sphere $x^2 + y^2 + z^2 = 9$. (15 points)

$\cos\varphi = \frac{1}{\sqrt{3}} \sin\varphi \quad \tan\varphi = \sqrt{3}$

$z = \frac{r}{\sqrt{3}} \qquad \tan^{-1}(\sqrt{3}) = \varphi = \pi/3$





$$\int_0^{2\pi}\int_0^{\pi/3}\int_0^3 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\frac{1}{3}\rho^3 \Big|_0^3 = 9$$

$$\int_0^{2\pi}\int_0^{\pi/3} 9 \sin\varphi \, d\varphi \, d\theta = \int_0^{2\pi} -9\cos\varphi \Big|_0^{\pi/3} d\theta$$

$$\int_0^{2\pi}\left(9 - \frac{9}{2}\right) d\theta = \left(\frac{9}{2}\right) 2\pi =$$

$$\boxed{9\pi}$$

alt:

$\frac{r}{\sqrt{3}} = \sqrt{9-r^2}$

$\frac{r^2}{3} = 9 - r^2 \Rightarrow r^2 = 27 - 3r^2$

$4r^2 = 27 \quad r = \frac{3\sqrt{3}}{2} = \sqrt{\frac{27}{4}}$

$u = 9 - r^2 \quad -\frac{1}{2}du = r\,dr$

$du = -2r\,dr$

$$\int_0^{2\pi}\int_0^{\frac{3\sqrt{3}}{2}}\int_{r/\sqrt{3}}^{\sqrt{9-r^2}} r \, dz \, dr \, d\theta = \int_0^{2\pi}\int_0^{\frac{3\sqrt{3}}{2}} rz\Big|_{\frac{r}{\sqrt{3}}}^{\sqrt{9-r^2}} dr\,d\theta = \int_0^{2\pi}\int_0^{\frac{3\sqrt{3}}{2}} r\sqrt{9-r^2} - \frac{r^2}{\sqrt{3}} \, dr\,d\theta$$

$$= \int_0^{2\pi} \left(\cancel{\tfrac{2}{3}}\right)\left(\cancel{-\tfrac{1}{2}}\right)(9-r^2)^{3/2} - \frac{r^3}{3\sqrt{3}}\Big|_0^{\frac{3\sqrt{3}}{2}} d\theta = \int_0^{2\pi} -\frac{1}{3}\left(9 - \frac{27}{4}\right)^{3/2} + \frac{1}{3}(9)^{3/2} - \frac{(3\sqrt{3})^3}{3\sqrt{3}\,8} \, d\theta = \int_0^{2\pi} -\frac{9}{8} + 9 - \frac{\overset{27}{\cancel{81\sqrt{3}}}}{\cancel{3\sqrt{3}}\,8} d\theta$$

$$= \int_0^{2\pi} 9/2 \, d\theta = 9\pi$$