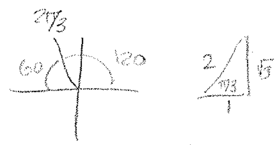
Math 2153 Homework #1 Key

1) a) $\|\vec{v}\| = 5,\ \theta = 120^\circ$

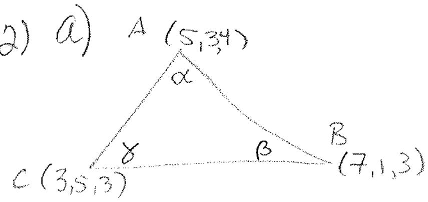


$$\vec{v} = 5\cos(2\pi/3)\hat{i} + 5\sin(2\pi/3)\hat{j} = -5\cos(\pi/3)\hat{i} + 5\sin(\pi/3)\hat{j} =$$
$$-5(1/2)\hat{i} + 5\left(\tfrac{\sqrt{3}}{2}\right)\hat{j} = -\tfrac{5}{2}\hat{i} + \tfrac{5\sqrt{3}}{2}\hat{j}$$

b) $\|\vec{v}\| = 8,\ \theta = -3.5$ radians

$$\vec{v} = 8\cos(-3.5)\hat{i} + 8\sin(-3.5)\hat{j} \approx -7.49\hat{i} + 2.81\hat{j}$$

2) a)



$\vec{AC} = \langle -2, 2, -1\rangle$ $\quad \vec{CA} = \langle 2, -2, 1\rangle$

$\vec{AB} = \langle 2, -2, -1\rangle$ $\quad \vec{BA} = \langle -2, 2, 1\rangle$

$\vec{BC} = \langle -4, 4, 0\rangle$ $\quad \vec{CB} = \langle 4, -4, 0\rangle$

$\vec{AC}\cdot\vec{AB} = \langle -2,2,-1\rangle\cdot\langle 2,-2,-1\rangle = -4-4+1 = -7$ obtuse

$\vec{BA}\cdot\vec{BC} = \langle -2,2,1\rangle\cdot\langle -4,4,0\rangle = 8+8+0 = 16$ acute

$\vec{CA}\cdot\vec{CB} = \langle 2,-2,1\rangle\cdot\langle 4,-4,0\rangle = 8+8+0 = 16$ acute

$\|\vec{BA}\| = \|\vec{AB}\| = \sqrt{2^2+2^2+1^2} = \sqrt{4+4+1} = \sqrt{9} = 3$

$\|\vec{AC}\| = \|\vec{CA}\| = \sqrt{2^2+2^2+1^2} = \sqrt{4+4+1} = \sqrt{9} = 3$

$\|\vec{BC}\| = \|\vec{CB}\| = \sqrt{4^2+4^2+0^2} = \sqrt{16+16+0} = \sqrt{32} = 4\sqrt{2}$

$\cos\alpha = \dfrac{-7}{3\cdot 3} = \dfrac{-7}{9}$ $\quad \cos\beta = \dfrac{16}{3\cdot 4\sqrt{2}} = \dfrac{4}{3\sqrt{2}}$ $\quad \cos\gamma = \dfrac{16}{3\cdot 4\sqrt{2}} = \dfrac{4}{3\sqrt{2}}$

$\alpha \approx 2.46$ rad. $\quad \beta \approx .3398$ rad. $\quad \gamma \approx .3398$ rad.

i) $\beta = \gamma$ $\therefore$ the triangle is isosceles

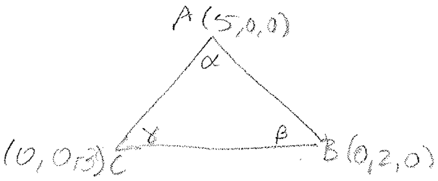
ii) $\overline{BC}$ is the longest side

midpoint of $\overline{BC} = \left(\frac{3+7}{2}, \frac{5+1}{2}, \frac{3+3}{2}\right) = \left(\frac{10}{2}, \frac{6}{2}, \frac{6}{2}\right) = (5,3,3)$

iii) obtuse triangle since $\alpha > \pi/2$

iv) $A = \frac{1}{2}\|\vec{AC}\times\vec{AB}\| = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & -1 \\ 2 & -2 & -1 \end{vmatrix} = \frac{1}{2}\|(-2-2)\hat{i} - (2-4)\hat{j} + (4-4)\hat{k}\| =$
$\frac{1}{2}\|\langle -4, 2, 0\rangle\| = \frac{1}{2}\sqrt{16+4} = \frac{1}{2}\sqrt{20} =$
$\frac{1}{2}2\sqrt{5} = \sqrt{5}$

2b)



$\vec{AC} = \langle -5,0,-3 \rangle$ $\quad$ $\vec{CA} = \langle 5,0,3 \rangle$

$\vec{AB} = \langle -5,2,0 \rangle$ $\quad$ $\vec{BA} = \langle 5,-2,0 \rangle$

$\vec{BC} = \langle 0,-2,-3 \rangle$ $\quad$ $\vec{CB} = \langle 0,2,3 \rangle$

i) $\vec{AB} \cdot \vec{AC} = \langle -5,2,0 \rangle \cdot \langle -5,0,-3 \rangle = 25+0+0 = 25$ acute

$\vec{BC} \cdot \vec{BA} = \langle 0,-2,-3 \rangle \cdot \langle 5,-2,0 \rangle = 0+4+0 = 4$ acute

$\vec{CA} \cdot \vec{CB} = \langle 5,0,3 \rangle \cdot \langle 0,2,3 \rangle = 0+0+9 = 9$ acute

neither

ii) $\|\vec{AC}\| = \|\vec{CA}\| = \sqrt{25+0+9} = \sqrt{34}$ longest side

$\|\vec{AB}\| = \|\vec{BA}\| = \sqrt{25+4+0} = \sqrt{29}$

$\|\vec{BC}\| = \|\vec{CB}\| = \sqrt{0+4+9} = \sqrt{13}$

midpoint of $\overline{AC} = \left(\frac{5+0}{2}, \frac{0+0}{2}, \frac{0-3}{2}\right) = \left(\frac{5}{2}, 0, \frac{-3}{2}\right)$

iii) acute

$\cos\alpha = \frac{25}{\sqrt{34}\sqrt{29}}$ $\quad$ $\cos\beta = \frac{4}{\sqrt{29}\cdot\sqrt{13}}$ $\quad$ $\cos\gamma = \frac{9}{\sqrt{34}\sqrt{13}}$

$\alpha \approx .6499$ rad. $\quad$ $\beta \approx 1.36$ rad $\quad$ $\gamma \approx 1.128$ rad.

all $< \pi/2$

iv) $A = \frac{1}{2}\|\vec{AC} \times \vec{AB}\| =$

$$\frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & 0 & -3 \\ -5 & 2 & 0 \end{vmatrix} = \frac{1}{2}\|\langle 0+6, -(0-15), -10-0 \rangle\| = \frac{1}{2}\|\langle 6,15,-10 \rangle\|$$

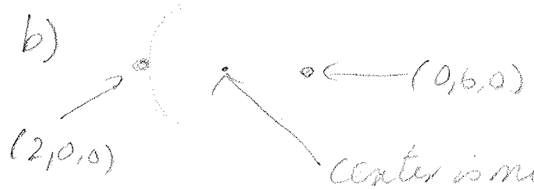
$$\frac{1}{2}\sqrt{36+225+100} = \frac{1}{2}\sqrt{361}$$

3 a) $x^2+9x+\frac{81}{4}+y^2-2y+1+z^2+10z+25 = -19+\frac{81}{4}+1+25$

$(x+\frac{9}{2})^2+(y-1)^2+(z+5)^2 = \frac{109}{4}$

center $(-9/2, 1, -5)$ $\quad$ radius $= \frac{\sqrt{109}}{2}$

3 b)



center is midpoint of diameter

$$\left(\frac{2+0}{2}, \frac{0+6}{2}, \frac{0+0}{2}\right) = (1,3,0) \text{ center}$$

distance from center to diameter is radius

$$r = \sqrt{(2-1)^2+(0-3)^2+0^2} = \sqrt{1+9+0} = \sqrt{10} = \text{radius}$$

$$(x-1)^2+(y-3)^2+z^2 = 10$$

4. the direction angle are, for a vector $\vec{v} = \langle a,b,c \rangle$

$$\cos\alpha = \frac{a}{\|\vec{v}\|}, \quad \cos\beta = \frac{b}{\|\vec{v}\|}, \quad \cos\gamma = \frac{c}{\|\vec{v}\|}$$

a) $\vec{u} = -4\hat{i}+3\hat{j}+5\hat{k}$ $\quad \|\vec{u}\| = \sqrt{16+9+25} = \sqrt{50} = 5\sqrt{2}$

$$\cos\alpha = \frac{-4}{5\sqrt{2}} \qquad \cos\beta = \frac{3}{5\sqrt{2}} \qquad \cos\gamma = \frac{\cancel{5}}{\cancel{5}\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\alpha \approx 2.172$ rad $(>\pi/2)$ $\qquad \beta \approx 1.13$ rad. $\qquad \gamma = \pi/4$

b) $\langle -2,6,1 \rangle$ $\quad \|\langle -2,6,1\rangle\| = \sqrt{4+36+1} = \sqrt{41}$

$$\cos\alpha = \frac{-2}{\sqrt{41}} \qquad \cos\beta = \frac{6}{\sqrt{41}} \qquad \cos\gamma = \frac{1}{\sqrt{41}}$$

$\alpha \approx 1.888$ rad $(>\pi/2)$ $\qquad \beta \approx .3567$ rad $\qquad \gamma \approx 1.414$ rad.

c) $A(-4,3,1)$, $B(-5,3,0)$

$\vec{AB} = \langle -5+4, 3-3, 0-1 \rangle = \langle -1,0,-1 \rangle$ $\quad \|\vec{AB}\| = \sqrt{1+0+1} = \sqrt{2}$

$$\cos\alpha = \frac{-1}{\sqrt{2}} \qquad \cos\beta = 0 \qquad \cos\gamma = \frac{-1}{\sqrt{2}}$$

$\alpha = 3\pi/4 \qquad \beta = \pi/2 \qquad \gamma = 3\pi/4$

5. prove that $|\vec{u}\cdot\vec{v}| \le \|\vec{u}\|\|\vec{v}\|$

$$|\vec{u}\cdot\vec{v}| = |\ \|\vec{u}\|\|\vec{v}\|\cos\theta| = \|\vec{u}\|\|\vec{v}\|\cdot|\cos\theta|$$

Since $\|\vec{u}\|, \|\vec{v}\|$ both positive

Since $|\cos\theta| \le 1$

this implies that

$$|\vec{u}\cdot\vec{v}| \le \|\vec{u}\|\|\vec{v}\|$$

6. $$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -8 & -6 & 4 \\ 10 & -12 & -2 \end{vmatrix} = (12+48)\hat{i} - (16-40)\hat{j} + (96+60)\hat{k}$$

$$= \langle 60, 24, 156 \rangle$$

Scale by 1/12 to make #'s easier to work w/

$$12\langle 5, 2, 13 \rangle$$

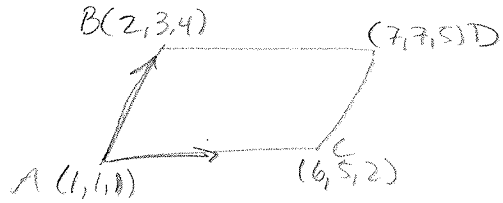
this vector or any multiple of it is orthogonal to the original two

make this a unit vector

$$\|\langle 5, 2, 13\rangle\| = \sqrt{25+4+169} = \sqrt{198} = 3\sqrt{22}$$

$$\vec{u} = \left\langle \frac{5}{3\sqrt{22}}, \frac{2}{3\sqrt{22}}, \frac{13}{3\sqrt{22}} \right\rangle$$

7. $A(1,1,1)$ $B(2,3,4)$ $C(6,5,2)$, $D(7,7,5)$



$$\vec{AB} = \langle 1, 2, 3 \rangle$$
$$\vec{AC} = \langle 5, 4, 1 \rangle$$

$$A = \|\vec{AB}\times\vec{AC}\| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 5 & 4 & 1 \end{vmatrix} = \|\langle 2-12, -(1-15), 4-10 \rangle\| =$$

$$\|\langle -10, 14, -6 \rangle\| =$$

$$\sqrt{100+196+36} = \sqrt{332}$$

$$= 2\sqrt{83}$$

8. $V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} = \langle 4+1, 0, 0 \rangle = \langle 5, 0, 0 \rangle$$

$$\langle 2,1,0 \rangle \cdot \langle 5,0,0 \rangle = 10 + 0 + 0 = 10$$

9. a) $P(5,-3,-4) \quad \perp \langle 2,-1,3 \rangle$

$$2(x-5) - 3(y+1) - 4(z-3) = 0$$

b) $P(-6,0,8) \quad \perp \quad x = 5-2t,\ y = -4+2t,\ z = 0$

$$\langle -2, 2, 0 \rangle$$

$$-2(x+6) + 2(y-0) + 0(z-8) = 0$$

$$-2(x+6) + 2y = 0$$

c) $P(2,3,-2)$, $Q(3,4,2)$, $R(-1,-1,0)$

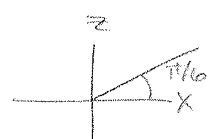
$\vec{PQ} = \langle 1,1,4 \rangle \qquad \vec{QR} = \langle -4,-5,-2 \rangle$

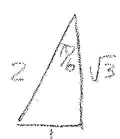
$$\vec{PQ} \times \vec{QR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 4 \\ -4 & -5 & -2 \end{vmatrix} = (-2+20)\hat{i} - (-2+16)\hat{j} + (-5+4)\hat{k}$$

$$\langle 18, -14, -1 \rangle$$

$$18(x-2) - 14(y-3) - 1(z+2) = 0$$

d) one vector in the plane $\hat{j}$





Second vector $\cos\theta\,\hat{i} + \sin\theta\,\hat{k}$

$= \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{k}$

$$\perp\text{vector} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{vmatrix} = \left(\tfrac{1}{2} - 0\right)\hat{i} - 0\hat{j} + \left(0 - \tfrac{\sqrt{3}}{2}\right)\hat{k}$$

$$\langle 1/2, 0, -\tfrac{\sqrt{3}}{2} \rangle$$

point $(0,0,0)$ since y-axis is goes through it

$$\tfrac{1}{2}(x-0) + 0(y-0) - \tfrac{\sqrt{3}}{2}(z-0) = 0$$

$$\tfrac{1}{2}x = \tfrac{\sqrt{3}}{2}z \implies x = \sqrt{3}z$$

9 e) vectors in plane

$\langle -2, 1, 1 \rangle$ and $\langle -3, 4, -1 \rangle$

$\perp$ vector $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 1 \\ -3 & 4 & -1 \end{vmatrix} = (-1-4)\hat{i} - (2+3)\hat{j} + (-8+3)\hat{k}$

$\langle -5, -5, -5 \rangle \underset{\text{scalar}}{\Rightarrow} \langle 1, 1, 1 \rangle$

point in plane obtained from either line

$(1, 4, 0)$ or $(2, 1, 2)$

$(x-1) + (y-4) + (z-0) = 0$

f) $A(2,2,1)$ $B(-1,1,-1)$

$\vec{AB}$ in plane $\langle -3, -1 -2 \rangle$

plane vector $\langle 2, -3, 1 \rangle$

if our target plane is $\perp$ to this plane, then this vector is $\parallel$ to our target plane

$\perp$ vector $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = (-1-6)\hat{i} - (-3+4)\hat{j} + (9+2)\hat{k}$

$\langle -7, -1, 11 \rangle$

$-7(x-2) - 1(y-2) + 11(z-1) = 0$