Math 2153 Homework #3 Key

1) a) $w(t)\vec{r}(t) = w(t)x_r(t)\hat{\imath} + w(t)y_r(t)\hat{\jmath} + w(t)z_r(t)\hat{k}$

$D_t[w(t)\vec{r}(t)] = D_t[w(t)x_r(t)]\hat{\imath} + D_t[w(t)y_r(t)]\hat{\jmath} + D_t[w(t)z_r(t)]\hat{k}$

$= [w'(t)x_r(t) + w(t)x_r'(t)]\hat{\imath} + [w'(t)y_r(t) + w(t)y_r'(t)]\hat{\jmath} + [w'(t)z_r(t) + w(t)z_r'(t)]\hat{k}$

$= w'(t)x_r(t)\hat{\imath} + w'(t)y_r(t)\hat{\jmath} + w'(t)z_r(t)\hat{k} +$
$w(t)x_r'(t)\hat{\imath} + w(t)y_r'(t)\hat{\jmath} + w(t)z_r'(t)\hat{k}$

$= w'(t)\vec{r}(t) + w(t)\vec{r}\,'(t)$

b) $\vec{u}(t) \times \vec{v}(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} =$

$(y_u z_v - z_u y_v)\hat{\imath} - (x_u z_v - z_u x_v)\hat{\jmath} + (x_u y_v - y_u x_v)\hat{k}$
$= (y_u z_v - z_u y_v)\hat{\imath} + (z_u x_v - x_u z_v)\hat{\jmath} + (x_u y_v - y_u x_v)\hat{k}$

$D_t[\vec{u}(t) \times \vec{v}(t)] =$

$(y_u' z_v + y_u z_v' - z_u' y_v - z_u y_v')\hat{\imath} + (z_u' x_v + z_u x_v' - x_u' z_v - x_u z_v')\hat{\jmath} +$
$(x_u' y_v + x_u y_v' - y_u' x_v - y_u x_v')\hat{k}$

$=$
$(y_u' z_v - z_u y_v)\hat{\imath} - (x_u' z_v - z_u' x_v)\hat{\jmath} + (x_u' y_v - y_u' x_v)\hat{k} +$
$(y_u z_v' - z_u y_v')\hat{\imath} - (x_u z_v' - z_u x_v')\hat{\jmath} + (x_u y_v' - y_u x_v')\hat{k}$

$= \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$

Check $\vec{u}\,'(t) \times v(t)$

$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_u' & y_u' & z_u' \\ x_v & y_v & z_v \end{vmatrix} = (y_u' z_v - y_v z_u')\hat{\imath} - (x_u' z_v - x_v z_u')\hat{\jmath} + (x_u' y_v - x_v y_u')\hat{k}$

1) b) cont'd

$$\vec{u}(t) \times \vec{v}'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_u & y_u & z_u \\ x_v' & y_v' & z_v' \end{vmatrix} = (y_u z_v' - y_v' z_u)\hat{i} - (x_u z_v' - x_v' z_u)\hat{j} + (x_u x_v' - y_u x_v')\hat{k}$$

c) $$\vec{r}(t) \times \vec{r}'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_r & y_r & z_r \\ x_r' & y_r' & z_r' \end{vmatrix} =$$

$$(y_r z_r' - z_r y_r')\hat{i} - (x_r z_r' - x_r' z_r)\hat{j} + (x_r y_r' - x_r' y_r)\hat{k}$$

$$D_t[\vec{r}(t) \times \vec{r}'(t)] =$$

$$(\cancel{y_r' z_r'} + y_r z_r'' - \cancel{z_r' y_r'} - z_r y_r'')\hat{i} - (\cancel{x_r' z_r'} + x_r z_r'' - x_r'' z_r - \cancel{x_r' z_r'})\hat{j} +$$
$$(\cancel{x_r' y_r'} + x_r y_r'' - x_r'' y_r - \cancel{x_r' y_r'})\hat{k} =$$

$$(y_r z_r'' - z_r y_r'')\hat{i} - (x_r z_r'' - x_r'' z_r)\hat{j} + (x_r y_r'' - x_r'' y_r)\hat{k}$$

$$= \vec{r}(t) \times \vec{r}''(t)$$

check $$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_r & y_r & z_r \\ x_r'' & y_r'' & z_r'' \end{vmatrix} = (y_r z_r'' - z_r y_r'')\hat{i} - (x_r z_r'' - x_r'' z_r)\hat{j} + (x_r y_r'' - x_r'' y_r)\hat{k}$$

d) for this problem we are going to use the simpler rules that $D_t[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$ and the rule proved in 1b $D_t[\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times v(t) + \vec{u}(t) \times v'(t)$ (the former rule is proved in a proof handout)

$$\frac{d}{dt}\left[\vec{r}(t) \cdot [\vec{u}(t) \times \vec{v}(t)]\right] = r'(t) \cdot [\vec{u}(t) \times \vec{v}(t)] + \vec{r}(t) \cdot \frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)]$$

$$= \vec{r}'(t) \cdot [\vec{u}(t) \times \vec{v}(t)] + \vec{r}(t)[\vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)] =$$

1d) cont'd

$= \vec{r}\,'(t)\,[\vec{u}(t) \times \vec{v}(t)] + \vec{r}(t)\,[\vec{u}\,'(t) \times \vec{v}(t)] + \vec{r}(t)\,[\vec{u}(t) \times \vec{v}\,'(t)]$

2) a. $\vec{r}(t) = 6t\,\hat{i} - 7t^2\,\hat{j} + t^3\,\hat{k}$ $\quad a=1$

$\vec{r}\,'(t) = 6\,\hat{i} - 14t\,\hat{j} + 3t^2\,\hat{k}$

$\vec{r}\,''(t) = 0\,\hat{i} - 14\,\hat{j} + 6t\,\hat{k}$

$\int 6t\,dt\,\hat{i} - \int 7t^2\,dt\,\hat{j} + \int t^3\,dt\,\hat{k} =$

$(3t^2 + C_1)\hat{i} + (-\frac{7}{3}t^3 + C_2)\hat{j} + (\frac{1}{4}t^4 + C_3)\hat{k}$

$\int_0^1 \vec{r}(t)\,dt = 3t^2\Big|_0^1\hat{i} + (-\frac{7}{3}t^3\Big|_0^1\hat{j} + (\frac{1}{4}t^4\Big|_0^1\hat{k} = 3\hat{i} - \frac{7}{3}\hat{j} + \frac{1}{4}\hat{k}$

$\|\vec{r}\,'(t)\| = \sqrt{36 + 196t^2 + 9t^4} = (36 + 196t^2 + 9t^4)^{1/2}$

$D_t\|\vec{r}\,'(t)\| = \frac{1}{2}(36 + 196t^2 + 9t^4)^{-1/2}(392t + 36t^3) = \dfrac{196t + 18t^3}{\sqrt{36 + 196t^2 + 9t^4}}$

$\int_1^2 \sqrt{36 + 196t^2 + 9t^4}\,dt$ (you can do this numerically)

≈ 23.0008 $\quad$ if you feel brave, this can be done by trig substitution after completing the square

b) $\vec{r}(t) = 4t^{1/2}\,\hat{i} + t^{5/2}\,\hat{j} + 2\ln(t)\,\hat{k}$ $\quad a=4$

$r'(t) = 2t^{-1/2}\,\hat{i} + \frac{5}{2}t^{3/2}\,\hat{j} + \frac{2}{t}\,\hat{k} = \frac{2}{\sqrt{t}}\hat{i} + \frac{5}{2}t\sqrt{t}\,\hat{j} + \frac{2}{t}\hat{k}$

$r''(t) = t^{-3/2}\,\hat{i} + \frac{15}{4}t^{1/2}\,\hat{j} - \frac{2}{t^2}\,\hat{k} = \frac{1}{t\sqrt{t}}\hat{i} + \frac{15}{4}\sqrt{t}\,\hat{j} - \frac{2}{t^2}\hat{k}$

$\int 4t^{1/2}\,dt\,\hat{i} + \int t^{5/2}\,dt\,\hat{j} + \int 2\ln(t)\,dt\,\hat{k}$

$u = \ln t \quad dv = dt$

$du = \frac{1}{t} \quad v = t$

$(\frac{8}{3}t^{3/2} + C_1)\hat{i} + (\frac{2}{7}t^{7/2} + C_2)\hat{j} + (2t\ln t - t + C_3)\hat{k}$

$\int_0^4 \vec{r}(t)\,dt = \frac{8}{3}t^{3/2}\Big|_0^4\hat{i} + (\frac{2}{7}t^{7/2}\Big|_0^4 + \lim_{b\to 0}(2t\ln t - t\Big|_b^4 = \frac{64}{3}\hat{i} + \frac{256}{7}\hat{j} + (8\ln 4 - 4)\hat{k}$

$\lim_{b\to 0} 8\ln 4 - 4 - \overset{0}{\widehat{2b\ln(b)}} - \overset{0}{b}$ converges

$\lim_{b\to 0} b\ln b = \lim_{b\to 0}\dfrac{\ln b}{\frac{1}{b}} = \lim_{b\to 0}\dfrac{\frac{1}{b}}{-\frac{1}{b^2}}\cdot\dfrac{b^2}{b^2} = \lim_{b\to 0}\dfrac{-b}{1} = 0$

2 b) continued

$$\|r'(t)\| = \sqrt{(2t^{1/2})^2 + \left(\tfrac{5}{2}t^{3/2}\right)^2 + \left(\tfrac{2}{t}\right)^2} = \sqrt{4t + \tfrac{25}{4}t^3 + \tfrac{4}{t^2}} = \underline{\sqrt{16t^3+25t^5+16}}$$

$$D_t[\|r'(t)\|] = \frac{\cancel{\tfrac{1}{2}}(16t^3+25t^5+16)^{-1/2}(48t^2+125t^4)(\cancel{2}t) - 2\left(\sqrt{16t^3+25t^5+16}\right)}{4t^2}$$

$$= \frac{48t^3+125t^5 - 2(16t^3+25t^5+16)}{4t^2\sqrt{16t^3+25t^5+16}} = \frac{48t^3+125t^5-32t^3-50t^5-32}{4t^2\sqrt{16t^3+25t^5+16}}$$

$$= \frac{16t^3+75t^5-32}{4t^2\sqrt{16t^3+25t^5+16}}$$

$$\int_1^2 \|r'(t)\|dt = \int_1^2 \frac{\sqrt{16t^3+25t^5+16}}{2t}dt \quad \text{(numerically)} \approx 5.487$$

c) $\vec{r}(t) = (\sin t - t\cos t)\hat{i} + (\cos t + t\sin t)\hat{j} + t^3\hat{k}$ $\qquad a = \pi/2$

$\vec{r}'(t) = (\cancel{\cos t} - \cancel{\cos t} + t\sin t)\hat{i} + (\cancel{-\sin t} + \cancel{\sin t} + t\cos t)\hat{j} + 2t\hat{k}$

$= t\sin t\,\hat{i} + t\cos t\,\hat{j} + 2t\hat{k}$

$\vec{r}''(t) = (\sin t + t\cos t)\hat{i} + (\cos t - t\sin t)\hat{j} + 2\hat{k}$

$\int \vec{r}(t)\,dt = \int \sin t - t\cos t\,dt\,\hat{i} + \int \cos t + t\sin t\,dt\,\hat{j} + \int t^3 dt\,\hat{k}$

$u = t \quad dv = \cos t \qquad\qquad w = t \quad dv = \sin t$

$du = dt \quad v = \sin t \qquad\qquad dw = dt \quad v = -\cos t$

$= \left(-\cos t - t\sin t + \int \sin t\,dt\right)\hat{i} + \left(\sin t - t\cos t + \int \cos t\,dt\right)\hat{j} + \left(\tfrac{1}{4}t^4 + C_3\right)\hat{k}$

$= (\cancel{-\cos t} - t\sin t + \cancel{\cos t} + C_1)\hat{i} + (\cancel{\sin t} - t\cos t - \cancel{\sin t} + C_2)\hat{j} + \left(\tfrac{1}{4}t^4 + C_3\right)\hat{k}$

$= (-t\sin t + C_1)\hat{i} + (-t\cos t + C_2)\hat{j} + \left(\tfrac{1}{4}t^4 + C_3\right)\hat{k}$

$\int_0^{\pi/2} \vec{r}(t)\,dt = -t\sin t\Big|_0^{\pi/2}\hat{i} - t\cos t\Big|_0^{\pi/2}\hat{j} + \left(\tfrac{1}{4}t^4\right)\Big|_0^{\pi/2}\hat{k} =$

$-\pi/2\,\hat{i} + 0\hat{j} + \frac{\pi^4}{64}\hat{k}$

2 c) cont'd

$\|\vec{r}'(t)\| = \sqrt{t^2\sin^2 t + t^2\cos^2 t + 4t^2} = \sqrt{5t^2} = \sqrt{5}\,t$

$D_t[\|\vec{r}'(t)\|] = \sqrt{5}$

$\int_1^2 \sqrt{5}\,t\,dt = \frac{\sqrt{5}}{2}t^2\Big|_1^2 = \frac{\sqrt{5}}{2}(4-1) = \frac{3\sqrt{5}}{2}$

d) $\vec{r}(t) = e^t\hat{\imath} + \sec^2 t\,\hat{\jmath} + \frac{1}{t^2+1}\hat{k}$ $\quad a=1$

$-2t(t^2+1)^{-2}$

$\vec{r}'(t) = e^t\hat{\imath} + 2\sec^2 t\tan t\,\hat{\jmath} + \frac{-2t}{(t^2+1)^2}\hat{k}$

$\vec{r}''(t) = e^t\hat{\imath} + (4\sec^2 t\tan^2 t + 2\sec^4 t)\hat{\jmath} + (-2(t^2+1)^{-2} - 2t(t^2+1)^{-3}(-2)(2t))\hat{k}$

$\frac{-2(t^2+1)+8t^2}{(t^2+1)^3}$

$= e^t\hat{\imath} + 2\sec^2 t(2\tan^2 t + \sec^2 t)\hat{\jmath} + \frac{6t^2-2}{(t^2+1)^3}\hat{k}$

$\int \vec{r}(t)\,dt = \int e^t\,dt\,\hat{\imath} + \int \sec^2 t\,dt\,\hat{\jmath} + \int \frac{1}{t^2+1}\,dt\,\hat{k}$

$= (e^t + C_1)\hat{\imath} + (\tan t + C_2)\hat{\jmath} + (\arctan t + C_3)\hat{k}$

$\int_0^1 \vec{r}(t)\,dt = e^t\Big|_0^1\hat{\imath} + (\tan t\Big|_0^1\hat{\jmath} + (\arctan t\Big|_0^1\hat{k}$

$(e-1)\hat{\imath} + \tan(1)\hat{\jmath} + \pi/4\,\hat{k}$

$\|\vec{r}'(t)\| = \sqrt{e^{2t} + \sec^4 t + \frac{1}{(t^2+1)^2}} = \sqrt{(e^{2t}+\sec^4 t)(t^2+1)^2+1}\,/(t^2+1)$

$D_t[\|\vec{r}'(t)\|] = \frac{1}{\cancel{2}}\left(e^{2t} + \sec^4 t + \frac{1}{(t^2+1)^2}\right)^{-1/2}\left(\cancel{2}e^{2t} + \overset{2}{\cancel{4}}\sec^4 t\tan t - \frac{\overset{2}{\cancel{4}}t}{(t^2+1)^3}\right)$

$= \frac{e^{2t} + 2\sec^4 t\tan t - \frac{2t}{(t^2+1)^3}}{\sqrt{e^{2t} + \sec^4 t + \frac{1}{(t^2+1)^2}}} =$

$\frac{(t^2+1)^3(e^{2t}+2\sec^4 t\tan t) - 2t}{\frac{\sqrt{(e^{2t}+\sec^4 t)(t^2+1)-1}}{\cancel{t^2+1}}(t^2+1)^{3/2}} = \frac{(t^2+1)^3(e^{2t}+2\sec^4 t\tan t) - 2t}{(t^2+1)^2\sqrt{(e^{2t}+\sec^4 t)(t^2+1)-1}}$

2 d) cont'd

$$\int_1^2 \frac{\sqrt{(e^{2t} + \sec^4 t)(t^2+1)^2 + 1}}{t^2+1}\, dt$$

not defined at $\pi/2 \approx 1.57$ does not converge.

e) $\vec{r}(t) = \frac{1}{2}\hat{\imath} + \hat{\jmath} - \hat{k}$

$\vec{r}\,'(t) = \vec{0} = \vec{r}\,''(t)$

$\int \vec{r}(t)\, dt = \int \frac{1}{2}\, dt\, \hat{\imath} + \int 1\, dt\, \hat{\jmath} + \int -1\, dt\, \hat{k}$

$= (\frac{1}{2}t + C_1)\hat{\imath} + (t + C_2)\hat{\jmath} + (-t + C_3)\hat{k}$

$\int_0^a r(t)\, dt = \frac{1}{2}a\hat{\imath} + a\hat{\jmath} - a\hat{k}$

$\|r'(t)\| = 0$

$D_t\left[\|r'(t)\|\right] = 0$

$\int_1^2 \|r'(t)\|\, dt = 0$

3) a. $\vec{r}(t) = t^2\hat{\imath} + t^3\hat{\jmath}$ $\quad (1,1) \quad t=1$

$\vec{r}\,'(t) = 2t\hat{\imath} + 3t^2\hat{\jmath}$ $\quad \vec{r}\,'(1) = 2\hat{\imath} + 3\hat{\jmath}$ velocity

$\vec{r}\,''(t) = 2\hat{\imath} + 6t\hat{\jmath}$ $\quad \vec{r}\,''(1) = 2\hat{\imath} + 6\hat{\jmath}$ acceleration

$\vec{r}\,'''(t) = 0\hat{\imath} + 6\hat{\jmath}$ $\quad \vec{r}\,'''(1) = 6\hat{\jmath}$ jerk

$\|\vec{r}\,'(t)\| = \sqrt{4t^2 + 9t^4}$ $\quad \|\vec{r}\,'(1)\| = \sqrt{4+9} = \sqrt{13}$ speed

b) $\vec{r}(t) = (t - \sin t)\hat{\imath} + (1 - \cos t)\hat{\jmath}$ $\quad (\pi, 2) \quad t=\pi$

$\vec{r}\,'(t) = (1 - \cos t)\hat{\imath} + \sin t\, \hat{\jmath}$ $\quad \vec{r}\,'(\pi) = 2\hat{\imath} + 0\hat{\jmath}$ velocity

$\vec{r}\,''(t) = \sin t\, \hat{\jmath} + \cos t\, \hat{\jmath}$ $\quad \vec{r}\,''(\pi) = 0\hat{\imath} - 1\hat{\jmath}$ acceleration

$\vec{r}\,'''(t) = \cos t\, \hat{\jmath} - \sin t\, \hat{\jmath}$ $\quad \vec{r}\,'''(\pi) = -1\hat{\imath} + 0\hat{\jmath}$ jerk

$\|\vec{r}\,'(t)\| = \sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2 - 2\cos t}$

$\|\vec{r}\,'(\pi)\| = \sqrt{2 - 2(-1)} = \sqrt{4} = 2$ speed

3) c. $\vec{r}(t) = 3t\hat{\imath} - t\hat{\jmath} + \frac{1}{4}t^2\hat{k}$

$\vec{r}\,'(t) = 3\hat{\imath} + 1\hat{\jmath} + \frac{1}{2}t\hat{k}$ velocity

$\vec{r}\,''(t) = 0\hat{\imath} + 0\hat{\jmath} + \frac{1}{2}\hat{k}$ acceleration

$\vec{r}\,'''(t) = \vec{0}$ jerk

$\|\vec{r}\,'(t)\| = \sqrt{9 + 1 + \frac{1}{4}t^2} = \frac{\sqrt{36+4+t^2}}{2} = \frac{\sqrt{40+t^2}}{2}$ speed

d. $\vec{r}(t) = e^t\cos t\,\hat{\imath} + e^t\sin t\,\hat{\jmath} + e^t\hat{k}$

$\vec{r}\,'(t) = (e^t\cos t - e^t\sin t)\hat{\imath} + (e^t\sin t + e^t\cos t)\hat{\jmath} + e^t\hat{k}$ velocity

$\vec{r}\,''(t) = (\cancel{e^t\cos t} - e^t\sin t - e^t\sin t - \cancel{e^t\cos t})\hat{\imath} + (\cancel{e^t\sin t} + e^t\cos t + e^t\cos t - \cancel{e^t\sin t})\hat{\jmath} + e^t\hat{k}$

$= -2e^t\sin t\,\hat{\imath} + e^t\cos t\,\hat{\jmath} + e^t\hat{k}$ acceleration

$\vec{r}\,'''(t) = (-2e^t\sin t - 2e^t\cos t)\hat{\imath} + (2e^t\cos t - 2e^t\sin t)\hat{\jmath} + e^t\hat{k}$ jerk

$\|\vec{r}\,'(t)\| = \sqrt{e^{2t}\cos^2 t - \cancel{2e^{2t}\sin t\cos t} + e^{2t}\sin^2 t + e^{2t}\sin^2 t + \cancel{2e^{2t}\sin t\cos t} + e^{2t}\cos^2 t + e^{2t}}$

$= \sqrt{2e^{2t}(\cos^2 t + \sin^2 t) + e^{2t}} = \sqrt{3e^{2t}} = \sqrt{3}e^t$ speed

4) review the first chapter on derivatives (differentials)

$\vec{r}(3.1) \approx \vec{r}(3) + \vec{r}\,'(3)(0.1)$

$\vec{r}(3) = 3\hat{\imath} + 4\hat{\jmath} + 4\hat{k}$

$\vec{r}\,'(t) = 1\hat{\imath} + \frac{1}{\cancel{2}}(25-t^2)^{-1/2}(\cancel{2}t)\hat{\jmath} + \frac{1}{\cancel{2}}(25-t^2)^{-1/2}(\cancel{2}t)\hat{k}$

$= 1\hat{\imath} + \frac{t}{\sqrt{25-t^2}}\hat{\jmath} + \frac{t}{\sqrt{25-t^2}}\hat{k}$

$\vec{r}\,'(3) = 1\hat{\imath} + \frac{3}{4}\hat{\jmath} + \frac{3}{4}\hat{k}$ tangent vector

$\vec{r}(3.1) = \langle 3, 4, 4\rangle + \langle 1, .75, .75\rangle(.1) =$

$\langle 3+.1, 4+.075, 4+.075\rangle = \langle 3.1, 4.075, 4.075\rangle$

5) a. $\vec{a}(t) = -\cos t\,\hat{\imath} - \sin t\,\hat{\jmath}$ $\quad \vec{v}(0) = \hat{\jmath} + \hat{k}$, $\vec{r}(0) = \hat{\imath}$

$\vec{v}(t) = \int -\cos t\,dt\,\hat{\imath} - \int \sin t\,dt\,\hat{\jmath} + \int 0\,dt\,\hat{k}$

$= (-\sin t + C_1)\hat{\imath} + (\cos t + C_2)\hat{\jmath} + C_3\hat{k}$

5a cont'd

$\vec{v}(0) = C_1\hat{i} + (1+C_2)\hat{j} + C_3\hat{k} = 0\hat{i} + 1\hat{j} + 1\hat{k}$

$C_1 = 0, \quad C_2 = 0, \quad C_3 = 1$

$\vec{v}(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$

$\vec{r}(t) = \int -\sin t\,dt\,\hat{i} + \int \cos t\,dt\,\hat{j} + \int 1\,dt\,\hat{k}$

$(\cos t + C_1)\hat{i} + (\sin t + C_2)\hat{j} + (t + C_3)\hat{k}$

$\vec{r}(0) = (1+C_1)\hat{i} + C_2\hat{j} + C_3\hat{k} = 1\hat{i} + 0\hat{j} + 0\hat{k}$

$C_1 = C_2 = C_3 = 0$

$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$ (helix)

b) $\vec{a}(t) = -32\hat{k} \quad \vec{v}(0) = 3\hat{i} - 2\hat{j} + \hat{k}, \quad \vec{r}(0) = 5\hat{j} + 2\hat{k}$

$\vec{v}(t) = \int 0\,dt\,\hat{i} + \int 0\,dt\,\hat{j} + \int -32\,dt\,\hat{k} =$

$C_1\hat{i} + C_2\hat{j} + (-32t + C_3)\hat{k}$

$\vec{v}(0) = C_1\hat{i} + C_2\hat{j} + C_3\hat{k} = 3\hat{i} - 2\hat{j} + \hat{k}$

$C_1 = 3, \quad C_2 = -2, \quad C_3 = 1$

$\vec{v}(t) = 3\hat{i} - 2\hat{j} + (-32t + 1)\hat{k}$

$\int \vec{v}(t)\,dt = \vec{r}(t) = \int 3\,dt\,\hat{i} - \int 2\,dt\,\hat{j} + \int (-32t + 1)\,dt\,\hat{k}$

$(3t + C_1)\hat{i} + (-2t + C_2)\hat{j} + (-16t^2 + t + C_3)\hat{k}$

$\vec{r}(0) = C_1\hat{i} + C_2\hat{j} + C_3\hat{k} = 0\hat{i} + 5\hat{j} + 2\hat{k}$

$C_1 = 0, \quad C_2 = 5, \quad C_3 = 2$

$\vec{r}(t) = 3t\,\hat{i} + (5 - 2t)\hat{j} + (-16t^2 + t + 2)\hat{k}$

c) $\vec{a}(t) = e^t\hat{i} - 8\hat{k} \quad \vec{v}(0) = 2\hat{i} + 3\hat{j} + \hat{k}, \quad \vec{r}(0) = \vec{0}$

$\vec{v}(t) = \int e^t\,dt\,\hat{i} + \int 0\,dt\,\hat{j} + \int -8\,dt\,\hat{k}$

$(e^t + C_1)\hat{i} + C_2\hat{j} + (8t + C_3)\hat{k}$

$\vec{v}(0) = (1 + C_1)\hat{i} + C_2\hat{j} + C_3\hat{k} = 2\hat{i} + 3\hat{j} + 1\hat{k}$

$C_1 = 1 \quad C_2 = 3, \quad C_3 = 1$

5 c) cont'd

$\vec{v}(t) = (e^t+1)\hat{\imath} + 3\hat{\jmath} + 1\hat{k}$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \int e^t+1\,dt\,\hat{\imath} + \int 3\,dt\,\hat{\jmath} + \int 1\,dt\,\hat{k}$

$(e^t+t+c_1)\hat{\imath} + (3t+c_2)\hat{\jmath} + (t+c_3)\hat{k}$

$\vec{r}(0) = (1+c_1)\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$

$c_1 = -1, \quad c_2 = 0, \quad c_3 = 0$

$\vec{r}(t) = (e^t+t-1)\hat{\imath} + 3t\hat{\jmath} + t\hat{k}$

6. $z_0 = 36{,}000$ feet

everything needs to be in feet & seconds since gravity is

$\downarrow a = g = -32\ \text{ft/sec}^2$

$\downarrow v_0 = 0$

let's assume for simplicity's sake that we define the path of the plane as the x-axis

$x'(0) = 600\text{ mph} = \frac{600\times 5280}{3600} = 880\text{ feet/sec}$

x_0 is the location dropped from

$\vec{r}(t) = \underset{x'(0)t + x_0}{(880t + x_0)}\hat{\imath} + \underset{-\frac{1}{2}gt^2 + v_0t + z_0}{(-16t^2+0t+36{,}000)}\hat{k}$

how long to hit ground?

$-16t^2 + 36{,}000 = 0$

$\frac{16t^2}{16} = \frac{36{,}000}{16} \Rightarrow t^2 = 2250 \quad t = 15\sqrt{10}$

needs to hit at $x=0$

$880(15\sqrt{10}) + x_0 = 0 \Rightarrow x_0 = -13{,}200\sqrt{10}$ feet away

or ≈ 7.9 miles before the target

7) a. $\vec{r}(t) = (t+1)\hat{\imath} + t^2\hat{\jmath} \quad [0,4]$

$\vec{r}\,'(t) = 1\hat{\imath} + 2t\hat{\jmath} \quad \|\vec{r}\,'(t)\| = \sqrt{1+4t^2}$

$\int_0^4 \sqrt{1+4t^2}\,dt$ to do this by hand need trig substitution and then the result needs to be done by parts

numerically $\approx 16.8186\ldots$

7 b) $\vec{r}(t) = a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath}$ $[0, 2\pi]$

$\vec{r}\,'(t) = -a\sin t\,\hat{\imath} + a\cos t\,\hat{\jmath}$

$\|\vec{r}\,'(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2} = a$

$\int_0^{2\pi} a\,dt = at\Big|_0^{2\pi} = 2\pi a$

c) $\vec{r}(t) = \hat{\imath} + t^2\hat{\jmath} + t^3\hat{k}$ $[0, 2]$

$\vec{r}\,'(t) = 0\hat{\imath} + 2t\hat{\jmath} + 3t^2\hat{k}$

$\|\vec{r}\,'(t)\| = \sqrt{4t^2 + 9t^4} = \sqrt{t^2(4+9t^2)} = t\sqrt{4+9t^2}$

$\int_0^2 t\sqrt{4+9t^2}\,dt$ $u = 4+9t^2$

$du = 18t\,dt \rightarrow \frac{1}{18}du = t\,dt$

$\int \frac{1}{18}u^{1/2}\,du = \frac{1}{\cancel{18}_{9}}\cdot\frac{\cancel{2}}{3}u^{3/2} \Rightarrow \frac{1}{27}(4+9t^2)^{3/2}\Big|_0^2 = \frac{1}{27}[40^{3/2} - 8]$

d) $\vec{r}(t) = (\cos t + t\sin t)\hat{\imath} + (\sin t - t\cos t)\hat{\jmath} + t^2\hat{k}$ $[0, \pi/2]$

$\vec{r}\,'(t) = [\cancel{-\sin t} + \cancel{\sin t} + t\cos t]\hat{\imath} + [\cancel{\cos t} - \cancel{\cos t} + t\sin t)\hat{\jmath} + 2t\hat{k}$

$= t\cos t\,\hat{\imath} + t\sin t\,\hat{\jmath} + 2t\hat{k}$

$\|\vec{r}\,'(t)\| = \sqrt{t^2\cos^2 t + t^2\sin^2 t + 4t^2} = \sqrt{t^2 + 4t^2} = \sqrt{5t^2} = \sqrt{5}\,t$

$\int_0^{\pi/2} \sqrt{5}\,t\,dt = \frac{5}{2}t^2\Big|_0^{\pi/2} = \frac{5}{2}\left(\frac{\pi}{2}\right)^2 = \frac{5\pi^2}{8}$

8) a. $\vec{r}(s) = (3+s)\hat{\imath} + \hat{\jmath}$ $s = 0$

$\vec{r}\,'(s) = 1\hat{\imath} + 0\hat{\jmath}$ $\|r'(s)\| = 1$ $\vec{T}(s) = \frac{1\hat{\imath} + 0\hat{\jmath}}{1} = \hat{\imath}$

$r''(s) = \vec{0}$ $T'(s) = \vec{0}$ $\|T'(s)\| = 0 = K$

$\|r''(s)\| = 0 = K$

the curvature is 0 because $\vec{r}(s)$ is a straight line

b) $\vec{r}(s) = (1 - \frac{1}{\sqrt{2}}s)\hat{\imath} + (1 + \frac{1}{\sqrt{2}})\hat{\jmath}$

$\vec{r}\,'(s) = -\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}$ $\|r'(s)\| = 1$ $\vec{T}(s) = \frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}$ $\vec{T}'(s) = \vec{0}$

$K = \|T'(s)\| = 0$

$r''(s) = \vec{0}$ $\|r''(s)\| = 0 = K$ also a line at all pts

8 c) $\vec{r}(t) = 4(\sin t - t\cos t)\hat{\imath} + 4(\cos t + t\sin t)\hat{\jmath} + \frac{2}{3}t^2\hat{k}$ $t=0$

$\vec{r}'(t) = 4(\cancel{\cos t} - \cancel{\cos t} + t\sin t)\hat{\imath} + 4(\cancel{-\sin t} + \cancel{\sin t} + t\cos t)\hat{\jmath} + \frac{4}{3}t\hat{k}$

$= 4t\sin t\,\hat{\imath} + 4t\cos t\,\hat{\jmath} + \frac{4}{3}t\,\hat{k}$

$\|r'(t)\| = \sqrt{16t^2\sin^2 t + 16t^2\cos^2 t + \frac{16}{3}t^2} = \sqrt{(16 + \frac{16}{3})t^2} = \sqrt{\frac{64}{3}t^2} = \frac{8t}{\sqrt{3}}$

$\vec{T}(t) = \frac{\sqrt{3}}{2}\sin t\,\hat{\imath} + \frac{\sqrt{3}}{2}\cos t\,\hat{\jmath} + \frac{\sqrt{3}}{2}\hat{k}$

$\vec{T}'(t) = \frac{\sqrt{3}}{2}\cos t\,\hat{\imath} - \frac{\sqrt{3}}{2}\sin t\,\hat{\jmath} + 0\hat{k}$ $\quad \|\vec{T}'(t)\| = \sqrt{\frac{3}{4}\cos^2 t + \frac{3}{4}\sin^2 t} = \frac{\sqrt{3}}{2}$

$K = \dfrac{\frac{\sqrt{3}}{2}}{\frac{8t}{\sqrt{3}}} = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{8t} = \frac{3}{16t}$ not defined at $t=0$

$\vec{r}''(t) = (4\sin t + 4t\cos t)\hat{\imath} + (4\cos t - 4t\sin t)\hat{\jmath} + \frac{4}{3}\hat{k}$

$$\vec{r}'\times\vec{r}'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4t\sin t & 4t\cos t & 4/3\,t \\ 4\sin t + 4t\cos t & 4\cos t - 4t\sin t & \frac{4}{3} \end{vmatrix} =$$

$(\cancel{\frac{16}{3}t\cos t} - \cancel{\frac{16}{3}t\cos t} + \frac{16}{3}t^2\sin t)\hat{\imath} - (\cancel{\frac{16}{3}t\sin t} - \cancel{\frac{16}{3}t\sin t} - \frac{16}{3}t^2\cos t)\hat{\jmath} +$
$(16t\cancel{\sin t}\cos t - 16t^2\sin^2 t - 16\cancel{t\sin t}\cos t - 16t^2\cos^2 t)\hat{k}$

$\frac{16}{3}t^2\sin t\,\hat{\imath} + \frac{16}{3}t^2\cos t\,\hat{\jmath} - 16t^2\hat{k}$

$\|r'(t)\times r''(t)\| = \sqrt{\frac{256}{9}t^4\sin^2 t + \frac{256}{9}t^4\cos^2 t + 256t^4} = \sqrt{(\frac{256}{9} + 256)t^4} =$

$\sqrt{\frac{2560}{9}t^4} = \frac{16\sqrt{10}}{3}t^2$

$K = \dfrac{\frac{16\sqrt{10}}{3}t^2}{\left(\frac{8t}{\sqrt{3}}\right)^3} = \frac{\cancel{16}\sqrt{10}}{\cancel{3}}\cancel{t^2}\cdot\frac{\cancel{3}\sqrt{3}}{8^{\cancel{3}}t^{\cancel{3}}} = \frac{\sqrt{30}}{32t}$ not defined at $t=0$

8·4

Radius of Curvature $= \frac{1}{K} = 0$

this point could represent a cusp in the graph

8) d. $\vec{r}(t) = 5\cos t\,\hat{i} + 4\cos t\,\hat{j}$ $t = \pi/3$

$\vec{r}'(t) = -5\sin t\,\hat{i} - 4\sin t\,\hat{j}$ $\|\vec{r}'(t)\| = \sqrt{25\sin^2 t + 16\sin^2 t} = \sqrt{41}\sin t$

$\vec{r}''(t) = -5\cos t\,\hat{i} - 4\cos t\,\hat{j}$

$$r' \times r'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5\sin t & -4\sin t & 0 \\ -5\cos t & -4\cos t & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + (20\sin t\cos t - 20\sin t\cos t)\hat{k}$$
$$= \vec{0}$$

$\|r'(t) \times \vec{r}''(t)\| = 0 = K$

$\vec{T}(t) = \dfrac{-5\cancel{\sin t}\,\hat{i} - 4\cancel{\sin t}\,\hat{j}}{\sqrt{41}\ \cancel{\sin t}} = \dfrac{-5}{\sqrt{41}}\hat{i} - \dfrac{4}{\sqrt{41}}\hat{j}$

$\vec{T}'(t) = \vec{0}$ $\|\vec{T}'(t)\| = 0 = K$ This is also a line at all points

e) $\vec{r}(t) = e^t\cos t\,\hat{i} + e^t\sin t\,\hat{j} + e^t\hat{k}$, $t = \pi$

$r'(t) = (e^t\cos t - e^t\sin t)\hat{i} + (e^t\sin t + e^t\cos t)\hat{j} + e^t\hat{k}$

$r''(t) = (\cancel{e^t\cos t} - e^t\sin t - e^t\sin t - \cancel{e^t\cos t})\hat{i} + (\cancel{e^t\sin t} + e^t\cos t + e^t\cos t - \cancel{e^t\sin t})\hat{j} + e^t\hat{k}$

$= (-2e^t\sin t)\hat{i} + 2e^t\cos t\,\hat{j} + e^t\hat{k}$

$$r' \times r'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ e^t\cos t - e^t\sin t & e^t\sin t + e^t\cos t & e^t \\ -2e^t\sin t & 2e^t\cos t & e^t \end{vmatrix} =$$

$(e^{2t}\sin t + e^{2t}\cos t - 2e^{2t}\cos t)\hat{i} - (e^{2t}\cos t - e^{2t}\sin t + 2e^{2t}\sin t)\hat{j} +$
$+ (2e^{2t}\cos^2 t - \cancel{2e^{2t}\sin t\cos t} + 2e^{2t}\sin^2 t + \cancel{2e^{2t}\sin t\cos t})\hat{k}$

$= (e^{2t}\sin t - e^{2t}\cos t)\hat{i} - (e^{2t}\cos t + e^{2t}\sin t)\hat{j} + (2e^{2t})\hat{k}$

$\|\vec{r}'(t) \times r''(t)\| = \sqrt{e^{4t}\sin^2 t - \cancel{2e^{4t}\sin t\cos t} + e^{4t}\cos^2 t + e^{4t}\cos^2 t + \cancel{2e^{4t}\sin t\cos t} + e^{4t}\sin^2 t + 4e^{4t}}$

$= \sqrt{2e^{4t} + 4e^{4t}} = \sqrt{6e^{4t}} = \sqrt{6}\,e^{2t}$

8e) cont'd

$$\|r'(t)\| = \sqrt{e^{2t}\cos^2 t - \cancel{2e^{2t}\sin t\cos t} + e^{2t}\sin^2 t + e^{2t}\sin^2 t + \cancel{2e^{2t}\sin t\cos t} + e^{2t}\cos^2 t + e^{2t}}$$

$$= \sqrt{2e^{2t} + e^{2t}} = \sqrt{3e^{2t}} = \sqrt{3}e^{t}$$

$$K = \frac{\sqrt{6}\,e^{2t}}{(\sqrt{3}\,e^{t})^3} = \frac{\sqrt{2}}{3}e^{-t}$$

$$\vec{T}(t) = \frac{(\cancel{e^t}\cos t - \cancel{e^t}\sin t)\hat{i} + (\cancel{e^t}\sin t + \cancel{e^t}\cos t)\hat{j} + \cancel{e^t}\hat{k}}{\sqrt{3}\cancel{e^t}} =$$

$$= \frac{(\cos t - \sin t)}{\sqrt{3}}\hat{i} + \frac{(\sin t + \cos t)}{\sqrt{3}}\hat{j} + \frac{\hat{k}}{\sqrt{3}}$$

$$\vec{T}'(t) = \frac{(-\sin t - \cos t)}{\sqrt{3}}\hat{i} + \frac{(\cos t - \sin t)}{\sqrt{3}}\hat{j} + 0\hat{k}$$

$$\|T'(t)\| = \sqrt{\tfrac{1}{3}(\sin^2 t + \cancel{2\sin t\cos t} + \cos^2 t) + \tfrac{1}{3}(\cos^2 t - \cancel{2\sin t\cos t} + \sin^2 t)} = \sqrt{\tfrac{1}{3}+\tfrac{1}{3}} = \sqrt{\tfrac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$

$$K = \frac{\frac{\sqrt{2}}{\sqrt{3}}}{\sqrt{3}e^{t}} = \frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}e^{t}} = \frac{\sqrt{2}}{3}e^{-t}$$

f) $$y' = \frac{3}{4}\cdot\frac{1}{\cancel{2}}(16-x^2)^{-1/2}(-\cancel{2}x) = \frac{-3x}{4\sqrt{16-x^2}}$$

$$y'' = \frac{3}{4}(-1)(16-x^2)^{-1/2} - \frac{3}{4}(-x)\left(\frac{-\cancel{1}}{\cancel{2}}\right)(16-x^2)^{-3/2}(\cancel{-2}x) =$$

$$\frac{16-x^2}{16-x^2}\cdot\frac{3}{4\sqrt{16-x^2}} + \frac{3x^2}{4(16-x^2)^{3/2}} = \frac{\overset{12}{\cancel{48}} - \cancel{3x^2} + \cancel{3x^2}}{\cancel{4}(16-x^2)^{3/2}} = \frac{12}{(16-x^2)^{3/2}}$$

$$K = \frac{\frac{12}{(16-x^2)^{3/2}}}{\left[1+\left(\frac{-3x}{4\sqrt{16-x^2}}\right)^2\right]^3} = \frac{\frac{12}{(16-x^2)^{3/2}}}{\left[1+\frac{9x^2}{16(16-x^2)}\right]^{3/2}} = \frac{\frac{12}{(16-x^2)^{3/2}}}{\sqrt{\frac{16(16-x^2)+9x^2}{16(16-x^2)^3}}} = \frac{12}{\cancel{(16-x^2)^{3/2}}}\cdot\frac{64\cancel{(16-x^2)^{3/2}}}{\sqrt{256-7x^2}}$$

$$= \frac{768}{\sqrt{256-7x^2}} = K \qquad \text{at } x=0 \quad \frac{768}{16} = 48$$

radius of curvature = $1/48$

8 f) cont'd

$$y^2 = \frac{9}{16}(16 - x^2) \Rightarrow y^2 = 9 - \frac{x^2}{16} \Rightarrow \frac{9x^2}{16} + y^2 = 9 \Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$$

ellipse $\vec{r}(t) = 4\cos t\,\hat{\imath} + 3\sin t\,\hat{\jmath}$

$t = \pi/2 \Leftrightarrow x = 0$

$$r'(t) = -4\sin t\,\hat{\imath} + 3\cos t\,\hat{\jmath}$$

$$r''(t) = -4\cos t\,\hat{\jmath} - 3\sin t\,\hat{\jmath}$$

$$r' \times r'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -4\sin t & 3\cos t & 0 \\ -4\cos t & -3\sin t & 0 \end{vmatrix} = (0)\hat{\imath} + (0)\hat{\jmath} + (12\sin^2 t + 12\cos^2 t)\hat{k}$$

$$= 12\hat{k}$$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = 12$$

$$\|\vec{r}'(t)\| = \sqrt{16\sin^2 t + 9\cos^2 t} = \sqrt{7\sin^2 t + 9\sin^2 t + 9\cos^2 t} = \sqrt{7\sin^2 t + 9}$$

$$K = \frac{12}{\left(\sqrt{7\sin^2 t + 9}\right)^3} \quad t = \pi/2 \Rightarrow \frac{12}{\left(\sqrt{7+9}\right)^3} = \frac{12}{4^3} = \frac{\cancel{12}^{\,3}}{\cancel{64}_{\,16}} = \frac{3}{16}$$

radius of curvature $= \frac{16}{3}$

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various methods for calculating curvature should agree.

bonus point for finding my error on 8c & 8f