Math 2153 Homework #4 Key

1)a. $y = x^3$

$y' = 3x^2,\ y'' = 6x \qquad K = \dfrac{|6x|}{[1+(3x^2)^2]^{3/2}} = \dfrac{|6x|}{[1+9x^4]^{3/2}}$

to calculate max find $K' = 0$ (this will be max or min, but if part of graph $\to K=0$ then this will be max)

$$K' = \frac{6(1+9x^4)^{3/2} - \frac{3}{2}(1+9x^4)^{1/2}(36x^3)6x}{(1+9x^4)^3} = 0$$

$$\underbrace{6(1+9x^4)^{1/2}}_{\neq 0}[1+9x^4-54x^4] = 0$$

$$[1-45x^4] = 0 \Rightarrow 45x^4 = 1 \Rightarrow \sqrt[4]{x^4} = \sqrt[4]{\tfrac{1}{45}} \Rightarrow x = \pm\sqrt[4]{\tfrac{1}{45}}$$

$x \approx \pm .386$

in the limit as $x \to \infty$, $K \to 0$ (degree of denom > deg of numerator)

at $x = 0$, $K = 0$

b) $y = e^x,\ y' = e^x,\ y'' = e^x$

$$K = \frac{e^x}{[1+e^{2x}]^{3/2}} \qquad K' = \frac{e^x(1+e^{2x})^{3/2} - \frac{3}{\cancel{2}}(1+e^{2x})^{1/2}(\cancel{2}e^{2x})e^x}{[1+e^{2x}]^3}$$

$$\underbrace{e^x}_{\neq 0}\underbrace{(1+e^{2x})^{1/2}}_{\neq 0}[1+e^{2x}-3e^{2x}] = 0$$

$$1-2e^{2x} = 0 \Rightarrow 2e^{2x} = 1 \Rightarrow e^{2x} = \tfrac{1}{2}$$

$$\Rightarrow \frac{2x}{2} = \frac{\ln\frac{1}{2}}{2} \Rightarrow x = \tfrac{1}{2}\ln\tfrac{1}{2} \Rightarrow x = \ln\left(\tfrac{1}{\sqrt{2}}\right)$$

in the limit as $x \to \infty$, $K \to 0$ since $\dfrac{e^x}{e^{3x}} \to 0$

Curvature is never exactly 0 since $e^x \neq 0$

3 d) $z = e^y \sin(xy)$

$z_x = e^y \cos(xy) \cdot y$ $\quad\quad z_x = e^1 \cos(-1)$

$z_y = e^y \sin(xy) + e^y \cos(xy) \cdot x$ $\quad z_y = e^1 \sin(-1) + e^1 \cos(-1)(-1)$

$= e(\sin(-1) - \cos(-1))$

$e^y \cos(xy) \cdot y = 0$ $\quad e^y \neq 0$ $\quad y\cos(xy) = 0$ $\quad y = 0$ or $\cos(xy) = 0$

$e^y(\sin(xy) + x\cos(xy)) = 0$

if $y=0$ $\quad \underset{0}{\sin(0)} + x\cos(xy) = 0$ $\quad x = 0$ or $\cos(xy) = 0$ $\quad (0,0)$

if $\cos(xy) = 0$

$\sin(xy) + 0 = 0 \Rightarrow \sin(xy) = 0$

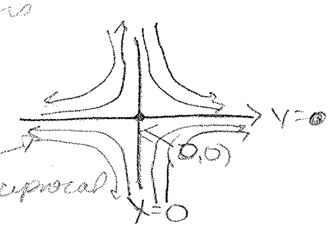
true if either $x=0$, or $y=0$

$\cos(xy) = 0$, if $xy = \frac{(2n+1)\pi}{2}$ $\quad (n \in \mathbb{Z})$ i.e. odd multiples of π

$y = \frac{(2n+1)\pi}{2x}$

to visualize these solutions

try looking at a 3D grapher to confirm results & interpret meaning



where constant $= \frac{(2n+1)\pi}{2}$

$y = \frac{(2n+1)\pi}{2} \cdot \frac{1}{x}$

both positive & negative values

e) $z = \operatorname{sech}(2x+3y)$

$z_x = \cosh(2x+3y) \cdot 2$ $\quad z_x(-1,1) = 2\cosh(1)$

$z_y = \cosh(2x+3y) \cdot 3$ $\quad z_y(-1,1) = 3\cosh(1)$

$\cosh(2x+3y) = 0$ $\quad$ but $\cosh(u) \neq 0$ for $u \in \mathbb{R}$

f) $f(x,y) = \tanh(xy^2)$

$f_x = \operatorname{sech}^2(xy^2) \cdot y^2$ $\quad f_x(-1,1) = \operatorname{sech}(-1)$ $\quad y^2 \operatorname{sech}(xy^2) = 0 \Rightarrow y = 0$

$f_y = \operatorname{sech}^2(xy^2)\, 2xy$ $\quad f_y(-1,1) = -2\operatorname{sech}(-1)$ $\quad 2xy \operatorname{sech}(xy^2) = 0 \Rightarrow x = 0$ or $y = 0$

$\operatorname{sech}(xy^2) \neq 0$

$(0,0)$

3 g) $f(x,y) = \arctan(y/x)$

$$f_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2} \qquad f_x(-1,1) = \frac{-1}{2}$$

$$f_y = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} \cdot \frac{x}{x} = \frac{x}{x^2+y^2} \qquad f_y(-1,1) = -\frac{1}{2}$$

$$\frac{-y}{x^2+y^2} = 0 \Rightarrow y = 0 \qquad \frac{x}{x^2+y^2} = 0 \Rightarrow x = 0$$

but derivatives are not defined at that point

h) $f(x,y) = x^2 + 4xy + y^2 - 4x + 16y + 3$

$f_x = 2x + 4y - 4 = 0 \qquad f_x(-1,1) = -2 + 4 - 4 = -2$

$f_y = 2y + 4x + 16 = 0 \qquad f_y(-1,1) = 2 - 4 + 16 = 14$

$$-2f_x = -\cancel{4x} - 8y + 8 = 0$$
$$\cancel{4x} + 2y + 16 = 0$$
$$\frac{-6y}{-6} = \frac{-24}{-6} \Rightarrow y = 4$$

$2x + 16 = 4 \Rightarrow 2x = -12$

$x = -6$

$(-6, 4)$

$8 - 24 + 16 = 0$ ✓

i) $f(x,y,z) = \dfrac{3xz}{x+y} = 3xz(x+y)^{-1}$

$$f_x = \frac{3z(x+y) - 1(3xz)}{(x+y)^2} = \frac{\cancel{3xz} + 3yz - \cancel{3xz}}{(x+y)^2} = \frac{3yz}{(x+y)^2}$$

$$f_y = 3xz(-1)(x+y)^{-2} = \frac{-3xz}{(x+y)^2}$$

$$f_z = \frac{3x}{x+y}$$

$$f_x(0,1,-2) = \frac{3(1)(-2)}{1} = -6$$
$$f_y(0,1,-2) = -\frac{3(0)(-2)}{1} = 0$$
$$f_z(0,1,-2) = \frac{3(0)}{1} = 0$$

$3yz = 0 \Rightarrow$

$y = 0$ or $z = 0$

$-3xz = 0 \Rightarrow$

$x = 0$ or $z = 0$

$3x = 0 \Rightarrow x = 0$

$(0, y, 0)$

if $y = 0$ & $x = 0$ derivatives are not defined

however if $y \neq 0$ all derivatives are still zero and well-defined when $x = 0$ and $z = 0$

3 j) $w = 3x^2y - 5xyz + 10yz^2$

$w_x = 6xy - 5yz \qquad w_x(0,1,-2) = 0 - 5(1)(-2) = 10$

$w_y = 3x^2 - 5xz + 10z^2 \qquad w_y(0,1,2) = 0 - 0 + 10(-2)^2 = 40$

$w_z = -5xy + 20yz \qquad w_z(0,1,2) = 0 + 20(1)(-2) = -40$

$6xy - 5yz = 0 \Rightarrow y(6x - 5z) = 0$ either $y = 0$ or $6x = 5z \Rightarrow x = \frac{5}{6}z$

$3x^2 - 5xz + 10z^2 = 0$

$-5xy + 20yz = 0 \Rightarrow y(-5x + 20z) = 0$ either $y = 0$ or $5x = 20z = x = 4z$

these can't both be true at the same time

$3x^2 - 5xz + 10z^2 = 0$ (there are no y's so try $x = 4z$)

$3(4z)^2 - 5(4z)z + 10z^2 = 0$

$48z^2 - 20z^2 + 10z^2 = 0$

$38z^2 = 0 \Rightarrow z = 0 \Rightarrow x = 0$ from $x = 4z$. & $y = 0$ since we still need $w_x = 0$

a similar calculation w/ $x = \frac{5}{6}z$ will produce similar results

thus $(0,0,0)$

k) $f(x,y,z) = (1 - x^2 - y^2 - z^2)^{-1/2}$

$f_x = -\frac{1}{2}(1 - x^2 - y^2 - z^2)^{-3/2}(-2x) = \frac{x}{(1-x^2-y^2-z^2)^{3/2}} \qquad f_x(0,1,-2) = 0$

$f_y = \frac{y}{(1-x^2-y^2-z^2)^{3/2}} \qquad f_y(0,1,-2) = \frac{1}{(1-0-1-4)^{3/2}} = \frac{1}{8}$

$f_z = \frac{z}{(1-x^2-y^2-z^2)^{3/2}} \qquad f_z(0,1,-2) = \frac{-2}{8} = -\frac{1}{4}$

$x = 0,\ y = 0,\ z = 0 \quad (0,0,0)$

4 a) $f_{xy} = 0 = f_{yx} = 0$

b) $z_{xy} = e^{x/y} \cdot \left(\frac{-x}{y^2}\right) + xe^{x/y}\left(\frac{-x}{y^2}\right)\frac{1}{y} + xe^{x/y}\left(\frac{-1}{y^2}\right)$ ✓

$z_{yx} = e^{x/y}\left(\frac{-x}{y^2}\right) + xe^{x/y}\left(\frac{1}{y}\right)\left(\frac{-x}{y^2}\right) + xe^{x/y}\left(\frac{-1}{y^2}\right)$

4c) $f_{xy} = \dfrac{(3y^2 - x^2)(x^2+y^2)^{\cancel{2}} - 2(\cancel{x^2+y^2})(2y)(y^3 - x^2y)}{(x^2+y^2)^{\cancel{4}3}} =$

$f_{yx} = \dfrac{(3x^2 - y^2)(x^2+y^2)^{\cancel{2}} - 2(\cancel{x^2+y^2})(2x)(x^3 - xy^2)}{(x^2+y^2)^{\cancel{4}3}} =$

$f_{xy} = \dfrac{3y^2x^2 + 3y^4 - x^4 - x^2y^2 - 4y^4 + 4x^2y^2}{(x^2+y^2)^3} = \dfrac{6x^2y^2 - y^4 - x^4}{(x^2+y^2)^3}$

$f_{yx} = \dfrac{3x^4 + 3x^2y^2 - x^2y^2 - y^4 - 4x^4 + 4x^2y^2}{(x^2+y^2)^3} = \dfrac{6x^2y^2 - y^4 - x^4}{(x^2+y^2)^3}$ ✓

d) $z_{xy} = e^y\cos(xy)\cdot y - e^y\sin(xy)xy + e^y\cos(xy)$ ✓

$z_{yx} = e^y\cos(xy)\cdot y + e^y\cos(xy) - e^y\sin(xy)xy$

e) $z_{xy} = 6\sinh(2x+3y)$

$z_{yz} = 6\sinh(2x+3y)$

f) $f_{xy} = 2\,\text{sech}^2(xy^2)\tanh(xy^2)\cdot 2xy\cdot y^2 + \text{sech}^2(xy^2)\,2y$ ✓

$f_{yx} = 2\,\text{sech}^2(xy^2)\tanh(xy^2)\cdot y^2\cdot 2xy + \text{sech}^2(xy^2)\cdot 2y$

g) $f_{xy} = \dfrac{-1(x^2+y^2) - 2y(-y)}{(x^2+y^2)^2} = \dfrac{-x^2 - y^2 + 2y^2}{(x^2+y^2)^2} = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$

$f_{yx} = \dfrac{1(x^2+y^2) - 2x(x)}{(x^2+y^2)^2} = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$ ✓

h) $f_{xy} = 4$ $\quad f_{yx} = 4$ ✓

i) $w_{xy} = 6x - 5z$ $\quad w_{xz} = -5y$ $\quad w_{xyz} = -5$ $\quad w_{xzy} = -5$

$w_{yx} = 6x - 5z$ $\quad w_{yz} = -5x + 20z$ $\quad w_{yxz} = -5$ $\quad w_{yzx} = -5$ ✓

$w_{zx} = -5y + 20y$ $\quad w_{zy} = -5x + 20y$ $\quad w_{zxy} = -5$ $\quad w_{zyx} = -5$

5) $f_x = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{(x+\Delta x)^2 - 3y^2 + 7 - (x^2 - 3y^2 + 7)}{\Delta x} =$

$\lim_{\Delta x \to 0} \dfrac{\cancel{x^2} + 2x\Delta x + \Delta x^2 - \cancel{3y^2} \cancel{+7} - \cancel{x^2} + \cancel{3y^2} - \cancel{7}}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{2x\Delta x + \Delta x^2}{\Delta x} =$

$\lim_{\Delta x \to 0} \dfrac{\cancel{\Delta x}(2x + \Delta x)}{\cancel{\Delta x}} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x$ (checks w/ 3a)

$f_y = \lim_{\Delta y \to 0} \dfrac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = \lim_{\Delta y \to 0} \dfrac{x^2 - 3(y+\Delta y)^2 + 7 - (x^2 - 3y^2 + 7)}{\Delta y} =$

$\lim_{\Delta y \to 0} \dfrac{\cancel{x^2} - \cancel{3y^2} - 6y\Delta y - 3\Delta y^2 \cancel{+7} - \cancel{x^2} + \cancel{3y^2} - \cancel{7}}{\Delta y} = \lim_{\Delta y \to 0} \dfrac{-6y\Delta y - 3\Delta y^2}{\Delta y} =$

$\lim_{\Delta y \to 0} \dfrac{\cancel{\Delta y}(-6y - 3\Delta y)}{\cancel{\Delta y}} = \lim_{\Delta y \to 0} -6y - 3\Delta y = -6y$ (checks w/ 3a)

6. a) $dz = f_x\,dx + f_y\,dy$ $\qquad f(x,y) = \dfrac{x^2}{y} = x^2y^{-1}$

$f_x = \dfrac{2x}{y} \qquad f_y = -\dfrac{x^2}{y^2} \qquad f(1,2) = \dfrac{1^2}{2} = \dfrac{1}{2}$

$f_x(1,2) = \dfrac{2(1)}{2} = 1 \quad f_y(1,2) = \dfrac{-1}{4} \qquad dx = .05 \quad dy = 0.1$

$dz = 1(.05) + \left(\dfrac{-1}{4}\right)(.1) = .025 = \dfrac{1}{40}$

$f(1.05, 2.1) \approx \dfrac{1}{2} + .025 = \dfrac{1}{2} + \dfrac{1}{40} = \dfrac{21}{40} = .525$

True value $f(1.05, 2.1) = 5.25$

b) $f(x,y) = xe^y \qquad f(1,2) = e^2$

$f_x = e^y \qquad f_x(1,2) = e^2$

$f_y = xe^y \qquad f_y(1,2) = e^2$

$dz = e^2(.05) + e^2(.1) = .15e^2$

$f(1.05, 2.1) \approx e^2 + .15e^2 = (1.15)e^2 \approx 8.4974\ldots$

true value $\approx 8.574\ldots$

6 c) $dw = g_x\,dx + g_y\,dy + g_z\,dz$

$g(x,y,z) = x^2yz^2 + \sin yz$ $\qquad f(1,2,0) = 1^2(2)(0) + \sin(2\cdot 0) = 0$

$g_x = 2xyz^2$ $\qquad g_x(1,2,0) = 0$ $\qquad dx = .05,\ dy = .1,\ dz = .01$

$g_y = x^2z^2 + \cos(yz)\cdot z$ $\qquad g_y(1,2,0) = 0$

$g_z = 2x^2yz + \cos(yz)\cdot y$ $\qquad g_z(1,2,0) = 0 + \cos(2\cdot 0)\cdot 2 = (1)(2) = 2$

$dw = (0)(.05) + (0)(.1) + 2(.01) = .02$

$g(1.05, 2.1, 0.01) \approx 0 + .02 = .02$

true value $\approx .02122998\ldots$

d) $g(x,y,z) = \dfrac{x+y}{z-2y} = (x+y)(z-2y)^{-1}$ $\qquad g(1,2,0) = \dfrac{1+2}{0-2(2)} = \dfrac{3}{-4} = -\dfrac{3}{4}$

$g_x = \dfrac{1}{z-2y}$ $\qquad g_x(1,2,0) = \dfrac{1}{-4}$

$g_y = \dfrac{1(z-2y) - (-2)(x+y)}{(z-2y)^2} = \dfrac{z - \cancel{2y} + 2x + \cancel{2y}}{(z-2y)^2} = \dfrac{z-2x}{(z-2y)^2}$ $\qquad g_y(1,2,0) = \dfrac{0-2}{(-4)^2} = \dfrac{-2}{16} = \dfrac{-1}{8}$

$g_z = (x+y)(-1)(z-2y)^{-2}(1) = \dfrac{-x-y}{(z-2y)^2}$ $\qquad g_z(1,2,0) = \dfrac{-1-2}{(-4)^2} = \dfrac{-3}{16}$

$g(1.05, 2.1, .01) \approx -\frac{3}{4} + (-\frac{1}{4})(.05) + (-\frac{1}{8})(.1) + (-\frac{3}{16})(.01) = \dfrac{-1243}{1600} = -.776875$

true value $= -.751789$

7) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$

a) $z = x^2 - y^2$ $\qquad z_x = 2x \quad z_{xx} = \dfrac{\partial^2 z}{\partial x^2} = 2$ $\qquad 2 + (-2) = 0$ ✓

$z_y = -2y \quad z_{yy} = \dfrac{\partial^2 z}{\partial y^2} = -2$

b) $z = \dfrac{y}{x^2+y^2}$ $\qquad z_x = y(-1)(x^2+y^2)^{-2}(2x) = \dfrac{-2xy}{(x^2+y^2)^2}$

$z_{xx} = \dfrac{\partial^2 z}{\partial x^2} = \dfrac{-2y(x^2+y^2)^{\cancel{2}} + \cancel{2(x^2+y^2)}\cdot 2x\,(+2xy)}{(x^2+y^2)^{\cancel{4}\,3}} =$

$= \dfrac{-2yx^2 - 2y^3 + 8x^2y}{(x^2+y^2)^3} = \dfrac{6x^2y - 2y^3}{(x^2+y^2)^3}$

7b cont'd

$$z_y = \frac{1(x^2+y^2) - 2y(y)}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$z_{yy} = \frac{\partial^2 z}{\partial x^2} = \frac{-2y(x^2+y^2)^{\cancel{2}} - 2\cancel{(x^2+y^2)}(2y)(x^2-y^2)}{(x^2+y^2)^{\cancel{4}3}} = \frac{-2x^2y - 2y^3 - 4x^2y + 4y^3}{(x^2+y^2)^3} =$$

$$\frac{-6x^2y + 2y^3}{(x^2+y^2)^3}$$

So $\frac{6x^2y - 2y^3}{(x^2+y^2)^3} + \frac{-6x^2y + 2y^3}{(x^2+y^2)^3} = 0$ ✓

c) $z = e^y \sin(x)$

$z_x = e^y \cos(x)$ $\quad z_{xx} = -e^y \sin(x)$

$z_y = e^y \sin(x)$ $\quad z_{yy} = e^y \sin(x)$

$-e^y \sin(x) + e^y \sin(x) = 0$ ✓

8) $dT = 1°$, $dv = 3$ mph

$C = 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}$

$C(T,v) = C(8,23)$ $\quad dC = C_T\,dT + C_v\,dv$ (error)

$C_T = .6215 + .4275v^{0.16}$

$C_v = -35.75(0.16)v^{-.84} + .4275T(0.16)v^{-.84} = \frac{-5.72 + .0684T}{v^{.84}}$

$C_T(8,23) = .6215 + .4275(23)^{.16} \approx 1.3275\ldots$

$C_v(8,23) = \frac{-5.72 + .0684(8)}{(23)^{.84}} = -.3714\ldots$

$dC = 1.3275(1) + (-.3714\ldots)(3) = -9.8145$

$|dC| \approx 9.8145$ propagated error or 9.815 w/4 signif digits

thus $C(8,23) = -12.681 \pm 9.815$