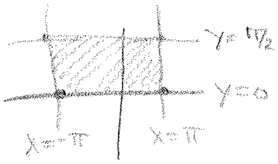
# Math 2153 Homework #6 Key

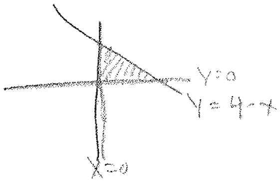
1) a. $\int_{-\pi}^{\pi}\int_0^{\pi/2} \sin x \sin y \, dy \, dx =$



$-\int_{-\pi}^{\pi} \sin x \cos y \Big|_0^{\pi/2} dx = -\int_{-\pi}^{\pi} \sin x (0-1)\, dx =$

$\int_{-\pi}^{\pi} \sin x \, dx = -\cos x \Big|_{-\pi}^{\pi} = -(-1)+(-1) = 0$

b. $\int_0^4\int_0^{4-x} x e^y \, dy \, dx = \int_0^4 x(e^{4-x}-1)\, dx$



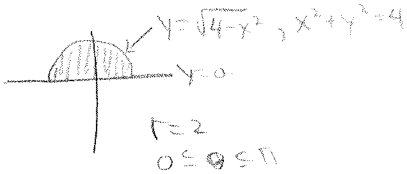
$= \int_0^4 x e^{4-x} - x \, dx$

|  | u | dv |
|---|---|---|
| + | x | $e^{4-x}$ |
| − | 1 | $-e^{4-x}$ |
|  | 0 | $e^{4-x}$ |

$-xe^{4-x} - e^{4-x} - \frac{1}{2}x^2 \Big|_0^4 =$

$-4e^0 - e^0 - 2 + 0 + e^4 + 0 = -4-1-2+e^4 = e^4-7$

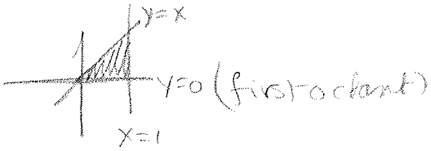
c. $\int_0^{\pi}\int_0^2 r^2 r \, dr \, d\theta = \int_0^{\pi}\int_0^2 r^3 \, dr \, d\theta =$



$r=2$
$0 \le \theta \le \pi$

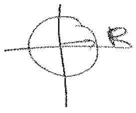
$\int_0^{\pi} \frac{1}{4}r^4 \Big|_0^2 d\theta = \int_0^{\pi} 4 \, d\theta = 4\pi$

2) a. $\int_0^1\int_0^x xy \, dy \, dx = \int_0^1 \frac{1}{2}xy^2 \Big|_0^x dx$



$= \int_0^1 \frac{1}{2}y^3 \, dx = \frac{1}{8}y^4 \Big|_0^1 = \frac{1}{8}$

b. $\int_0^{2\pi}\int_0^R 2\sqrt{R^2-r^2}\, r \, dr \, d\theta$ $\quad u = R^2-r^2$
$-du = 2r\,dr$



$Z = \pm\sqrt{R^2-x^2-y^2}$
$Z = \pm\sqrt{R^2-r^2}$

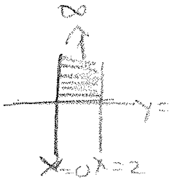
$-\int u^{1/2} du = -\frac{2}{3}u^{3/2}$

$-\int_0^{2\pi} \frac{2}{3}(R^2-r^2)^{3/2}\Big|_0^R d\theta = -\int_0^{2\pi}\frac{2}{3}\left((0)^{3/2}-R^3\right) d\theta =$

$\frac{2}{3}R^3\int_0^{2\pi} d\theta = \frac{2}{3}R^3 \cdot 2\pi = \frac{4}{3}\pi R^3$

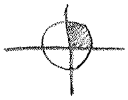
this can also be done easily in spherical
but for now, we'll stick to polar

2) c. $\int_0^2 \int_0^\infty \frac{1}{1+y^2}\,dy\,dx = \int_0^2 \arctan y \Big|_0^\infty dx$



$= \int_0^2 \pi/2\,dx = \pi$

3) a. $\int_0^{\pi/2} \int_0^1 r\cos\theta\, r\sin\theta\, dr\, d\theta =$



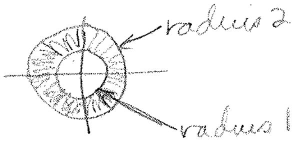
$\int_0^{\pi/2} \int_0^1 r^3 \cos\theta \sin\theta\, dr\, d\theta = \int_0^{\pi/2} \frac{1}{4} r^4 \Big|_0^1 \cos\theta \sin\theta\, d\theta = \int_0^{\pi/2} \frac{1}{4} \cos\theta \sin\theta\, d\theta$

$u = \sin\theta$
$du = \cos\theta\, d\theta$

$\frac{1}{8} \sin^2\theta \Big|_0^{\pi/2} = \frac{1}{8}$

b. $\int_0^{2\pi} \int_1^2 2(\ln r)\, r\, dr\, d\theta$

$u = \ln r \quad dv = r\,dr$
$du = \frac{1}{r} dr \quad v = \frac{1}{2} r^2.$

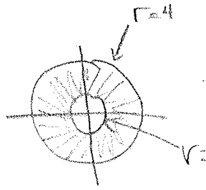


$\int_0^{2\pi} \left[ \frac{1}{2} r^2 \ln r - \int_1^2 \cancel{\frac{1}{r}} \cdot \frac{1}{2} r^2\, dr \right] d\theta = \int_0^{2\pi} \frac{1}{2} r^2 \ln r - \frac{1}{2}\frac{1}{2} r^2 \Big|_1^2 d\theta =$

$\int_0^{2\pi} 2\ln 2 - 1 - \frac{1}{2}(0) + \frac{1}{4}\, d\theta = \left[ 2\ln 2 - \frac{3}{4} \right] 2\pi = 4\pi \ln 2 - \frac{3\pi}{2}$

c. $\int_0^{2\pi} \int_1^4 \sqrt{16 - r^2}\, r\, dr\, d\theta$

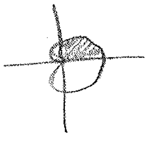
$u = 16 - r^2$
$-\frac{1}{2} du = r\,dr$



$-\frac{1}{2} \int u^{1/2}\, du \Rightarrow -\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} (16 - r^2)^{3/2} \Big|_1^4$

$-\frac{1}{3} \int_0^{2\pi} (0 - 15^{3/2})\, d\theta = \frac{1}{3} \cdot 15^{3/2} \cdot 2\pi = \frac{1}{\cancel{3}} \cancel{3}\sqrt{3} \cdot 5\sqrt{5} \cdot 2\pi = 10\sqrt{15}\,\pi$

d. $\int_0^{\pi/2} \int_0^{2+\cos\theta} (1 + \sin r)\, r\, dr\, d\theta =$



$\int_0^{\pi/2} \frac{1}{2} r^2 - r\cos r + \sin r \Big|_0^{2+\cos\theta} d\theta$

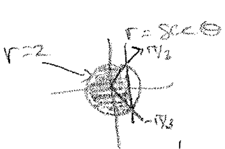
| | u | dv |
|---|---|---|
| + | r | sin r |
| − | 1 | −cos r |
| + | 0 | −sin r |

$\int_0^{\pi/2} \frac{1}{2}(2+\cos\theta)^2 - (2+\cos\theta)\left[\cos(2+\cos\theta)\right] - \sin(2+\cos\theta)\, d\theta$

these last 2 terms can only be done numerically

$\approx 8.3528\ldots$

e. $\int_{-\pi/3}^{\pi/3} \int_0^{\sec\theta} \sqrt{16 - 2r^2}\, r\, dr\, d\theta + \int_{\pi/3}^{5\pi/3} \int_0^2 \sqrt{16 - 2r^2}\, r\, dr\, d\theta$



$u = 16 - 2r^2$
$-\frac{1}{4} du = r\,dr$
$-\frac{1}{4} \int u^{1/2}\, du = -\frac{1}{\cancel{4}_2} \frac{\cancel{2}}{3} u^{3/2}$

$2 = \sec\theta \Rightarrow \frac{1}{2} = \cos\theta \quad \theta = \pm \pi/3$

$-\frac{2}{3} \int_{-\pi/3}^{\pi/3} (16 - 2r^2)^{3/2} \Big|_0^{\sec\theta} d\theta - \frac{2}{3} \int_{\pi/3}^{5\pi/3} (16 - 2r^2)^{3/2} \Big|_0^2 d\theta$

3 e) cont'd

$$-\frac{2}{3}\int_{-\pi/3}^{\pi/3}(16-2\sec^2\theta)^{3/2}-64\,d\theta-\frac{2}{3}\int_{\pi/3}^{5\pi/3}8^{3/2}-64\,d\theta=$$

This one must be done numerically

though trig sub might work

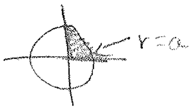
if $\sqrt{2}\sec\theta=4\sin\alpha$

$$\approx -\frac{2}{3}(95.22109493)+\frac{2}{3}(64)\left(\frac{2\pi}{3}\right)+\frac{2}{3}\left(64-8^{3/2}\right)\left(\frac{4\pi}{3}\right)=$$

$$\frac{2}{3}(2\pi)(64)-\frac{2}{3}(8)^{3/2}\left(\frac{4\pi}{3}\right)-\frac{2}{3}(95.22109\ldots)$$

$$\frac{256\pi}{3}-\frac{2}{3}\left(\frac{8\sqrt{2}\pi}{3}+95.22109\ldots\right)\approx 196.703\ldots$$

4) a. $M=\int_0^{\pi/2}\int_0^a kr^3\,dr\,d\theta=\int_0^{2\pi}\frac{k}{4}r^4\Big|_0^a d\theta=$



$$\frac{k}{4}a^4\cdot\frac{\pi}{2}=\frac{k\pi}{8}a^4$$

$$M_y=\int_0^{\pi/2}\int_0^a kr^2\,r\cos\theta\,r\,dr\,d\theta=\int_0^{\pi/2}\int_0^a kr^4\cos\theta\,dr\,d\theta=\int_0^{\pi/2}\frac{k}{5}r^5\Big|_0^a\cos\theta\,d\theta$$

$$=\int_0^{\pi/2}\frac{k}{5}a^5\cos\theta\,d\theta=\frac{k}{5}a^5\sin\theta\Big|_0^{\pi/2}=\frac{k}{5}a^5$$

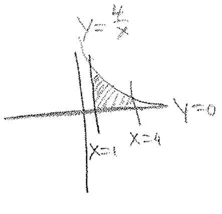
$$M_x=\int_0^{\pi/2}\int_0^a kr^2r\sin\theta\,r\,dr\,d\theta=\int_0^{\pi/2}\int_0^a kr^4\sin\theta\,dr\,d\theta=\int_0^{\pi/2}\frac{k}{5}r^5\Big|_0^a\sin\theta\,d\theta$$

$$\int_0^{\pi/2}\frac{k}{5}a^5\sin\theta\,d\theta=-\frac{k}{5}a^5\cos\theta\Big|_0^{\pi/2}=-\frac{k}{5}a^5(0-1)=\frac{k}{5}a^5$$

$$\bar{x}=\frac{M_y}{M}=\frac{\frac{\cancel{k}}{5}a^{\cancel{5}}}{\frac{\cancel{k}\pi}{8}\cancel{a^4}}=\frac{a}{5}\cdot\frac{8}{\pi}=\frac{8a}{5\pi}\qquad \bar{y}=\frac{M_x}{M}=\frac{\frac{\cancel{k}}{5}a^{\cancel{5}}}{\frac{\cancel{k}\pi}{8}\cancel{a^4}}=\frac{8a}{5\pi}$$

$(\bar{x},\bar{y})$ center of mass: $\left(\frac{8a}{5\pi},\frac{8a}{5\pi}\right)$

b. $M=\int_1^4\int_0^{4/x}kx^2\,dy\,dx=\int_1^4 kx^2y\Big|_0^{4/x}dx=\int_1^4 kx^2\left(\frac{4}{x}\right)dx=$



$$\int_1^4 4kx\,dx=2kx^2\Big|_1^4=32k-2k=30k$$

$$M_y=\int_1^4\int_0^{4/x}kx^3\,dy\,dx=\int_1^4 kx^3y\Big|_0^{4/x}dx=\int_1^4 kx^3\left(\frac{4}{x}\right)dx=\int_1^4 4kx^2\,dx=$$

$$\frac{4}{3}kx^3\Big|_1^4=\frac{256k}{3}-\frac{4k}{3}=\frac{252k}{3}=84k$$

4 b) cont'd

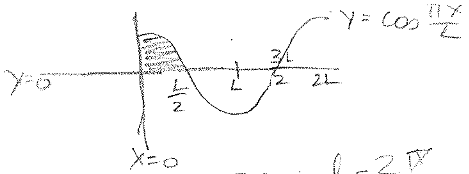
$M_x = \int_1^4 \int_0^{4/x} kx^2 y \, dy \, dx = \int_1^4 \frac{k}{2} x^2 y^2 \Big|_0^{4/x} dx = \int_1^4 \frac{k}{2} x^2 \left(\frac{4}{x}\right)^2 dx = \int_1^4 8k \, dx =$

$8kx \Big|_1^4 = 32k - 8k = 24k$

$\bar{x} = \frac{M_y}{M} = \frac{\overset{4}{\cancel{24}}\cancel{k}}{\underset{5}{\cancel{30}}\cancel{k}} = \frac{4}{5}$ $\quad \bar{y} = \frac{M_x}{M} = \frac{84\cancel{k}}{30\cancel{k}} = \frac{\overset{14}{\cancel{28}}}{\underset{5}{\cancel{10}}} = \frac{14}{5}$

$(\bar{x}, \bar{y}) = \left(\frac{4}{5}, \frac{14}{5}\right)$

c. $M = \int_0^{L/2} \int_0^{\cos(\frac{\pi x}{L})} k \, dy \, dx = \int_0^{L/2} k \cos\left(\frac{\pi x}{L}\right) dx$



$\frac{L}{\pi} k \sin\left(\frac{\pi x}{L}\right)\Big|_0^{L/2} = \frac{L}{\pi} k \sin\left(\frac{\pi (L/2)}{L}\right) = \frac{Lk}{\pi}$

period $= \frac{2\pi}{\frac{\pi}{L}} = \frac{L}{2}$

$M_y = \int_0^{L/2} \int_0^{\cos(\frac{\pi x}{L})} kx \, dy \, dx = \int_0^{L/2} kx \cos\left(\frac{\pi x}{L}\right) dx = k\left[\frac{Lx}{\pi} \sin\left(\frac{\pi x}{L}\right) + \frac{L^2}{\pi^2} \cos\left(\frac{\pi x}{L}\right)\right]_0^{L/2}$

$= k\left[\frac{L^2}{2\pi} - \frac{L^2}{\pi^2}\right] = \frac{kL^2(\pi - 2)}{2\pi^2}$

| | u | dv |
|---|---|---|
| + | $x$ | $\cos\frac{\pi x}{L}$ |
| − | $1$ | $\frac{L}{\pi}\sin\left(\frac{\pi x}{L}\right)$ |
| | $0$ | $-\frac{L^2}{\pi^2}\cos\left(\frac{\pi x}{L}\right)$ |

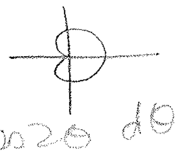
$M_x = \int_0^{L/2} \int_0^{\cos(\frac{\pi x}{L})} ky \, dy \, dx = \int_0^{L/2} \frac{k}{2} \cos^2\left(\frac{\pi x}{L}\right) dx =$

$\frac{k}{4} \int_0^{L/2} 1 + \cos\left(\frac{2\pi x}{L}\right) dx = \frac{k}{4}\left[x + \frac{L}{2\pi} \sin\left(\frac{2\pi x}{L}\right)\right]_0^{L/2} = \frac{k}{4}\left[\frac{L}{2}\right] = \frac{kL}{8}$

$\bar{x} = \frac{M_y}{M} = \frac{\cancel{k}\cancel{L}^{2}(\pi - 2)}{2\pi^{\cancel{2}}} \cdot \frac{\cancel{\pi}}{\cancel{L}\cancel{k}} = \frac{\pi - 2}{2\pi}$ $\quad \bar{y} = \frac{M_x}{M} = \frac{\cancel{k}\cancel{L}}{8} \cdot \frac{\pi}{\cancel{L}\cancel{k}} = \frac{\pi}{8}$

$(\bar{x}, \bar{y}) = \left(\frac{\pi - 2}{2\pi}, \frac{\pi}{8}\right)$

d. $M = \int_0^{2\pi} \int_0^{1+\cos\theta} k \, r \, dr \, d\theta = \frac{k}{2} \int_0^{2\pi} r^2 \Big|_0^{1+\cos\theta} d\theta =$



$\frac{k}{2} \int_0^{2\pi} 1 + 2\cos\theta + \underbrace{\cos^2\theta}_{\frac{1}{2} + \frac{1}{2}\cos 2\theta} \, d\theta = \frac{k}{2} \int_0^{\pi} \frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta \, d\theta =$

$\frac{k}{2}\left[\frac{3}{2}\theta + 2\overset{0}{\cancel{\sin\theta}} + \frac{1}{4}\overset{0}{\cancel{\sin 2\theta}}\right|_0^{2\pi} = \frac{k}{2} \cdot \frac{3}{\cancel{2}}(\cancel{2}\pi) = \frac{3k\pi}{2}$

$M_y = \int_0^{2\pi} \int_0^{1+\cos\theta} kr^2 \cos\theta \, dr \, d\theta = \int_0^{2\pi} \frac{k}{3} r^3 \Big|_0^{1+\cos\theta} \cos\theta \, d\theta = \frac{k}{3} \int_0^{2\pi} \cos\theta (1 + \cos\theta)^3 d\theta$

4d cont'd

$$= \frac{k}{3}\int_0^{2\pi} \cos\theta + 3\cos^2\theta + 3\cos^3\theta + \cos^4\theta \, d\theta =$$

$$\frac{k}{3}\int_0^{2\pi} \cos\theta + \frac{3}{2} + \frac{3}{2}\cos(2\theta) + 3\cos\theta(1-\sin^2\theta) + \frac{1}{4}(1+2\cos 2\theta+\underbrace{\cos^2 2\theta}_{\frac{1}{2}+\frac{1}{2}\cos 4\theta})\, d\theta$$

$$\frac{k}{3}\int_0^{2\pi} 4\cos\theta - 3\cos\theta\sin^2\theta + \frac{15}{8} + 2\cos 2\theta + \frac{1}{8}\cos 4\theta \, d\theta$$

$$\frac{k}{3}\left[4\overset{0}{\cancel{\sin\theta}} - \overset{0}{\cancel{\sin^3\theta}} + \frac{15}{8}\theta + \overset{0}{\cancel{\sin 2\theta}} + \frac{1}{32}\overset{0}{\cancel{\sin 4\theta}}\right]_0^{2\pi} = \frac{k\overset{5}{\cancel{15}}}{\cancel{3}8}\cdot 2\pi = \frac{5k\pi}{4}$$

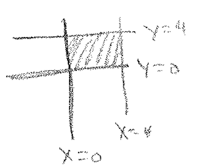
$$M_x = \int_0^{2\pi}\int_0^{1+\cos\theta} kr^2\sin\theta\, dr\, d\theta = \int_0^{2\pi}\frac{k}{3}r^3\Big|_0^{1+\cos\theta}\sin\theta\, dr\, d\theta = \int_0^{2\pi}\frac{k}{3}(1+\cos\theta)^3\sin\theta\, d\theta$$

$$= -\frac{k}{3}\left[\frac{(1+\cos\theta)^4}{4}\Big|_0^{2\pi} = -\frac{k}{3}\left[\frac{2^4}{4}-\frac{2^4}{4}\right] = 0\right.$$ (we expect this by symmetry)

$$\bar{x} = \frac{M_y}{M} = \frac{\cancel{k}5\cancel{\pi}}{\underset{2}{\cancel{4}}}\cdot\frac{\cancel{2}}{3\cancel{k\pi}} = \frac{5}{6} \qquad \bar{y} = \frac{0}{M} = 0$$

$$(\bar{x},\bar{y}) = \left(\frac{5}{6}, 0\right)$$

5) a. $M = \int_0^4\int_0^4\int_0^{4-x} ky\, dz\, dy\, dx = \int_0^4\int_0^4 kyz\Big|_0^{4-x} dy\, dx$



square in plane

$$\int_0^4\int_0^4 ky(4-x)\, dy\, dx = \int_0^4 \frac{k}{2}y^2(4-x)\Big|_0^4 dx =$$

$$\int_0^4 \frac{k}{2}(16)(4-x)\, dx = 8k\int_0^4 4-x\, dx = 8k\left(4x-\frac{1}{2}x^2\Big|_0^4 = 8k(16-8) = 64k\right.$$

$$M_{xy} = \int_0^4\int_0^4\int_0^{4-x} kyz\, dz\, dy\, dx = \frac{k}{2}\int_0^4\int_0^4 y(4-x)^2\, dy\, dx = \frac{k}{4}\int_0^4 y^2\Big|_0^4(4-x)^2\, dx$$

$$4k\int_0^4 16-8x+x^2\, dx = 4k\left(16x-4x^2+\frac{1}{3}x^3\Big|_0^4 = 4k\left(64-64+\frac{64}{3}\right) = \frac{256k}{3}\right.$$

$$M_{yz} = \int_0^4\int_0^4\int_0^{4-x} kyx\, dz\, dy\, dx = \int_0^4\int_0^4 kyx(4-x)\, dy\, dy = \int_0^4 \frac{k}{2}(16)(4x-x^2)\, dx$$

$$8k\left[2x^2-\frac{1}{3}x^3\Big|_0^4 = 8k\left[32-\frac{64}{3}\right] = \frac{256k}{3}\right.$$

$$M_{xz} = \int_0^4\int_0^4\int_0^{4-x} ky^2\, dz\, dy\, dx = \int_0^4\int_0^4 ky^2(4-x)\, dy\, dx = \int_0^4 \frac{k}{3}y^3\Big|_0^4(4-x)\, dx = \frac{64k}{3}\int_0^4 4-x\, dx$$

$$= \frac{64k}{3}\left[4x-\frac{1}{2}x^2\Big|_0^4 = \frac{64k}{3}[16-8] = \frac{512k}{3}\right.$$

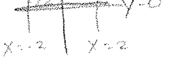
5 a) cont'd

$$\bar{X} = \frac{M_{yz}}{M} = \frac{\overset{4}{\cancel{256k}}}{3} \cdot \frac{1}{\cancel{64k}} = \frac{4}{3} \qquad \bar{Y} = \frac{M_{xz}}{M} = \frac{\overset{8}{\cancel{512k}}}{3} \cdot \frac{1}{\cancel{64k}} = \frac{8}{3} \qquad \bar{Z} = \frac{M_{xy}}{M} = \frac{\overset{4}{\cancel{256k}}}{3} \cdot \frac{1}{\cancel{64k}} = \frac{4}{3}$$

$(\bar{x}, \bar{y}, \bar{z})$ center of mass $\left(\frac{4}{3}, \frac{8}{3}, \frac{4}{3}\right)$

b)

$$M = \int_0^1 \int_{-2}^2 \int_0^{\frac{1}{y^2+1}} kz \, dz \, dx \, dy = \int_0^1 \int_{-2}^2 \frac{k}{2} z^2 \Big|_0^{\frac{1}{y^2+1}} dx \, dy = \frac{k}{2} \int_0^1 \int_{-2}^2 \frac{1}{(y^2+1)^2} \, dx \, dy$$



$$\frac{k}{2} \int_0^1 \frac{x}{(y^2+1)^2} \Big|_{-2}^2 dy = 2k \int_0^1 \frac{1}{(y^2+1)^2} \, dy$$

$y = \tan\theta$ $\quad y^2+1 = \tan^2\theta + 1 = \sec^2\theta$

$dy = \sec^2\theta \, d\theta$ $\quad y = 0 \Rightarrow \theta = 0$

$y = 1 \Rightarrow \theta = \pi/4$

$$2k \int_0^{\pi/4} \frac{\sec^2\theta \, d\theta}{(\sec^2\theta)^2} = 2k \int_0^{\pi/4} \frac{d\theta}{\sec^2\theta} = 2k \int_0^{\pi/4} \cos^2\theta \, d\theta = k \int_0^{\pi/4} 1 + \cos 2\theta \, d\theta =$$

$$k\left[\theta + \tfrac{1}{2}\sin 2\theta \Big|_0^{\pi/4} = k\left[\pi/4 + \tfrac{1}{2}\cdot(1)\right] = \frac{k(\pi+2)}{4}$$

$$M_{xy} = \int_0^1 \int_{-2}^2 \int_0^{\frac{1}{y^2+1}} kz^2 \, dz \, dx \, dy = \frac{k}{3} \int_0^1 \int_{-2}^2 \frac{1}{(y^2+1)^3} \, dx \, dy = \frac{4k}{3} \int_0^1 \frac{1}{(y^2+1)^3} \, dy$$

Same trig sub as above

$$\frac{4k}{3} \int_0^{\pi/4} \frac{\sec^2\theta \, d\theta}{(\sec^2\theta)^3} = \frac{4k}{3} \int_0^{\pi/4} \frac{d\theta}{\sec^4\theta} = \frac{4k}{3} \int_0^{\pi/4} \cos^4\theta \, d\theta = \frac{4k}{3} \cdot \frac{1}{4} \int_0^{\pi/4} (1+\cos 2\theta)^2 d\theta$$

$$= \frac{k}{3} \int_0^{\pi/4} 1 + 2\cos 2\theta + \underbrace{\cos^2 2\theta}_{\frac{1}{2}(1+\cos 4\theta)} \, d\theta = \frac{k}{3} \int_0^{\pi/4} \frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta \, d\theta =$$

$$\frac{k}{3}\left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\overset{0}{\cancel{\sin 4\theta}} \Big|_0^{\pi/4} = \frac{k}{3}\left[\frac{3\pi}{8} + 1\right] = \frac{k(3\pi+8)}{24}$$

$$M_{yz} = \int_0^1 \int_{-2}^2 \int_0^{\frac{1}{y^2+1}} kxz \, dz \, dx \, dy = \frac{k}{2} \int_0^1 \int_{-2}^2 \frac{x}{(y^2+1)^2} \, dx \, dy = \frac{k}{4} \int_0^1 \frac{x^2}{(y^2+1)^2} \Big|_2^2 dy = 0$$

$$M_{xz} = \int_0^1 \int_{-2}^2 \int_0^{\frac{1}{y^2+1}} kyz \, dz \, dx \, dy = \frac{k}{2} \int_0^1 \int_{-2}^2 \frac{y}{(y^2+1)^2} \, dx \, dy = \frac{k}{\cancel{4}} \int_0^1 \frac{\cancel{4}y}{(y^2+1)^2} dy =$$

$$k \int_0^1 \frac{y}{(y^2+1)^2} \, dy$$

$u = y^2+1$

$\frac{1}{2} du = \cancel{2} y \, dy$

$$\frac{k}{2} \int \frac{du}{u^2} = \frac{k}{2} \int u^{-2} du = -\frac{k}{2} \frac{1}{u} \Rightarrow -\frac{k}{2} \frac{1}{y^2+1} \Big|_0^1 =$$

$$-\frac{k}{2}\left[\frac{1}{2} - 1\right] = \frac{k}{2}\left[\frac{1}{2}\right] = \frac{k}{4}$$

$$\bar{X} = \frac{M_{yz}}{M} = \frac{0}{M} = 0 \qquad \bar{Y} = \frac{M_{xz}}{M} = \frac{\cancel{k}}{\cancel{4}} \cdot \frac{\cancel{4}}{\cancel{k}(\pi+2)} = \frac{1}{\pi+2} \qquad \bar{Z} = \frac{M_{xy}}{M} = \frac{\cancel{k}(3\pi+8)}{\underset{6}{\cancel{24}}} \cdot \frac{\cancel{4}}{\cancel{k}(\pi+2)} = \frac{3\pi+8}{6\pi+12}$$

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, \frac{1}{\pi+2}, \frac{3\pi+8}{6\pi+12}\right)$$

6. $I = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \qquad I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dy\, dx$ change to polar

$\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\, r\, dr\, d\theta \qquad \begin{matrix} u = -r^2/2 \\ -du = r\,dr \end{matrix} \qquad -\int e^u\, du = -e^u$

$\int_0^{2\pi} -e^{-r^2/2}\Big|_0^{\infty} d\theta = -\int_0^{2\pi}(0-1)\,d\theta = \int_0^{2\pi} d\theta = 2\pi$

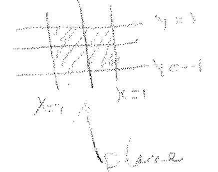
Since $I^2 = 2\pi$, $I = \sqrt{2\pi}$

7.a) $\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} x^2y^2z^2\, dx\, dy\, dz = \frac{1}{3}\int_{-1}^{1}\int_{-1}^{1} x^3\Big|_{-1}^{1} y^2z^2\, dy\, dz$

$\frac{2}{3}\int_{-1}^{1}\int_{-1}^{1} y^2z^2\, dy\, dz = \frac{2}{3}\int_{-1}^{1} y^2\, dy \cdot \int_{-1}^{1} z^2\, dz = \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} =$

$\frac{8}{27}$

region is a cube centered at origin w/ sides of length 2

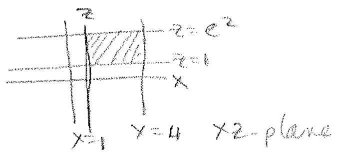


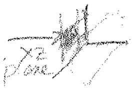
b) $\int_1^4\int_1^{e^2}\int_0^{1/xz} \ln z\, dy\, dz\, dx = \int_1^4\int_1^{e^2} \frac{\ln z}{xz}\, dz\, dx =$

$\int_1^4 \frac{1}{x}\int_1^{e^2} \frac{\ln z}{z}\, dz\, dx \qquad \begin{matrix} u = \ln z \\ du = \frac{1}{z}dz \end{matrix} \qquad \int u\, du = \frac{1}{2}u^2$

$\int_1^4 \frac{1}{x}\frac{(\ln z)^2}{2}\Big|_1^{e^2} dx = \frac{1}{2}\int_1^4 \frac{1}{x}(2^2-0)\, dx = 2\int_1^4 \frac{1}{x}\, dx = 2\ln x\Big|_1^4$

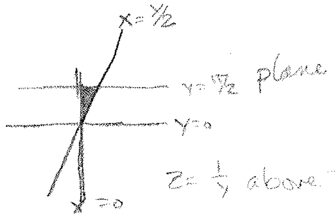
$= 2\ln 4$





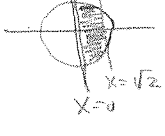
c) $\int_0^{\pi/2}\int_0^{y/2}\int_0^{1/y} \sin y\, dz\, dx\, dy = \int_0^{\pi/2}\int_0^{y/2} \frac{1}{y}\sin y\, dx\, dy =$

$\int_0^{\pi/2} \cancel{\frac{1}{y}}\sin y \cdot \frac{\cancel{y}}{2}\, dy = -\frac{\cos y}{2}\Big|_0^{\pi/2} = -\frac{1}{2}(0+1) = -\frac{1}{2}$



d) $\int_0^{\sqrt{2}}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{x^2} y\, dz\, dy\, dx = \int_0^{\sqrt{2}}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} yx^2\, dy\, dx =$

$\int_0^{\sqrt{2}} \frac{1}{2}y^2x^2\Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} = \int_0^{\sqrt{2}} \frac{1}{2}x^2\left[4-x^2-(4-x^2)\right]dx = 0$



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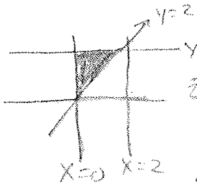
7) e. $\int_0^2\int_{2x}^4\int_0^{\sqrt{y^2-4x^2}} dz\,dy\,dx = \int_0^2\int_{2x}^4 \sqrt{y^2-4x^2}\,dy\,dx$

↑ $a = 2x$ trig sub



$z = \sqrt{y^2-4x^2}$ above (cone)

$z^2+4x^2-y^2=0$

elliptic cone around y-axis

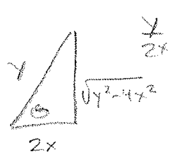
$y = 2x\sec\theta$

$\sqrt{4x^2\sec^2\theta-4x^2} = \sqrt{4x^2(\sec^2\theta-1)} =$

$\sqrt{4x^2\tan^2\theta} = 2x\tan\theta$

$dy = 2x\sec\theta\tan\theta$

$\frac{y}{2x} = \sec\theta$



$\int 2x\tan\theta \cdot 2x\sec\theta\tan\theta\,d\theta$

$4x^2\int(\overset{\tan^2\theta}{\sec^2\theta-1})\sec\theta\,d\theta =$

$4x^2\int \sec^3\theta - \sec\theta\,d\theta$

$\int\sec^3\theta\,d\theta =$

$u = \sec\theta$

$du = \sec\theta\tan\theta\,d\theta$

$dv = \sec^2\theta$

$v = \tan\theta$

$\sec\theta\tan\theta - \int\sec\theta\tan^2\theta\,d\theta$

$\sec\theta\tan\theta - \int\sec\theta(\sec^2\theta-1)\,d\theta =$

$\sec\theta\tan\theta - \int\sec^3\theta\,d\theta + \int\sec\theta\,d\theta = \int\sec^3\theta\,d\theta$
$+\int\sec^3\theta\,d\theta \qquad\qquad + \int\sec^3\theta\,d\theta$

$\frac{\sec\theta\tan\theta}{2} + \frac{\int\sec\theta\,d\theta}{2} = \frac{2\int\sec^3\theta\,d\theta}{2}$

$\int\sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int\sec\theta\,d\theta$

$4x^2\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int\sec\theta\,d\theta - \int\sec\theta\,d\theta\right]$

$4x^2\left[\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\int\sec\theta\,d\theta\right] =$

$\overset{2}{4}x^2\left[\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta|\right]$

$2x^2\left[\frac{y}{2x}\frac{\sqrt{y^2-4x^2}}{2x} - \ln\left|\frac{y}{2x}+\frac{\sqrt{y^2-4x^2}}{2x}\right|\right]_{2x}^{4}$

$2x^2\left[\frac{4}{2x}\frac{\sqrt{16-4x^2}}{2x} - \cancelto{0}{\frac{2x}{2x}\cdot\frac{\sqrt{4x^2-4x^2}}{2x}} - \ln\left|\frac{4}{2x}+\frac{\sqrt{16-4x^2}}{2x}\right| + \cancelto{0}{\ln\left|\underset{=1}{\frac{2x}{2x}}+\frac{\sqrt{4x^2-4x^2}}{2x}\right|}\right] =$

$2x^2\left[\frac{2}{x}\cdot\frac{2\sqrt{4-x^2}}{\cancel{2}x} - \ln\left|\frac{2}{x}+\frac{2\sqrt{4-x^2}}{2x}\right|\right] = 4\sqrt{4-x^2} - 2x^2\ln\left|\frac{2}{x}+\frac{\cancel{2}\sqrt{4-x^2}}{\cancel{2}x}\right|$

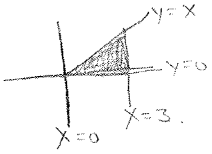
$\int_0^2 4\sqrt{4-x^2} - 2x^2\ln\left|\frac{2-\sqrt{4-x^2}}{\cancel{x}}\right|dx = \int_0^2 4\sqrt{4-x^2} - 2x^2\ln|2-\sqrt{4-x^2}| + 2x^2\ln x\,dx$

because of the middle term, do the last step numerically
(this should converge at zero since $\int x\ln x\,dx$ also converges at 0

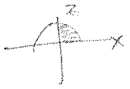
$\approx 16.755\ldots$

f. $\int_0^3\int_0^x\int_0^{9-x^2} dz\,dy\,dx = \int_0^3\int_0^x 9-x^2\,dy\,dx =$

$\int_0^3 x(9-x^2)\,dx = \int_0^3 9x - x^3\,dx = \frac{9}{2}x^2 - \frac{1}{4}x^4\Big|_0^3 = \frac{81}{2} - \frac{81}{4} = \frac{81}{4}$



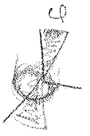
$z = 9-x^2$ above



g. $\int_0^{\pi/4}\int_0^2\int_0^{2-r} dz\,dr\,d\theta = \int_0^{\pi/4}\int_0^2 2-r\,dr\,d\theta = \int_0^{\pi/4} 2r - \frac{1}{2}r^2\Big|_0^2\,d\theta =$

$\int_0^{\pi/4} 4-2\,d\theta = \int_0^{\pi/4} 2\,d\theta = 2\cdot\frac{\pi}{4} = \frac{\pi}{2}$

7) h. $\int_0^{\pi/4}\int_0^{\pi/4}\int_0^{\cos\theta} \rho^2 \sin\varphi\cos\varphi\, d\rho\, d\theta\, d\varphi =$



$\rho = \cos\theta$

$\int_0^{\pi/4}\int_0^{\pi/4} \frac{1}{3}\rho^3\Big|_0^{\cos\theta} \sin\varphi\cos\varphi\, d\theta\, d\varphi = \frac13\int_0^{\pi/4}\int_0^{\pi/4}\cos^3\theta \sin\varphi\cos\varphi\, d\theta\, d\varphi =$

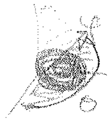
$\frac13\int_0^{\pi/4}\int_0^{\pi/4}\underbrace{\cos\theta(1-\sin^2\theta)}_{u=\sin\theta\ du=\cos\theta}\sin\varphi\cos\varphi\,d\theta\,d\varphi = \frac13\int_0^{\pi/4}\left(\sin\theta-\frac13\sin^3\theta\right)\Big|_0^{\pi/4}\sin\varphi\cos\varphi\,d\varphi$

$\int 1-u^2\,du = u-\frac{u^3}{3}$

$\frac13\int_0^{\pi/4}\left(\frac{1}{\sqrt2}-\frac13\cdot\frac{1}{2\sqrt2}\right)\sin\varphi\cos\varphi\,d\varphi = \frac13\left(\frac{6-1}{6\sqrt2}\right)\frac12\sin^2\varphi\Big|_0^{\pi/4} = \frac{5}{36\sqrt2}\left(\frac{1}{\sqrt2}\right)^2 = \frac{5}{72\sqrt2}$

$u=\sin\varphi$
$du=\cos\varphi$
$\int u\,du = \frac12u^2$

i) $\int_0^{\pi/2}\int_0^{\pi}\int_0^{\sin\theta} 2\cos\varphi\,\rho^2\,d\rho\,d\theta\,d\varphi = \int_0^{\pi/2}\int_0^{\pi}\frac23\rho^3\Big|_0^{\sin\theta}\cos\varphi\,d\theta\,d\varphi$



$\rho = \sin\theta$

$= \frac23\int_0^{\pi/2}\int_0^{\pi}\sin^3\theta\,\cos\varphi\,d\theta\,d\varphi = \frac23\int_0^{\pi/2}\int_0^{\pi}\sin\theta(1-\cos^2\theta)\cos\varphi\,d\theta\,d\varphi$

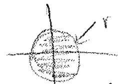
$u=\cos\theta$
$-du=\sin\theta\,d\theta$
$\int u^2-1 = \frac{u^3}{3}-u$

$\frac23\int_0^{\pi/2}\frac{\cos^3\theta}{3}-\cos\theta\Big|_0^{\pi}\cos\varphi\,d\varphi =$

$\frac23\int_0^{\pi/2}\left(\frac{(-1)^3}{3}-(-1)-\frac{1^3}{3}+1\right)\cos\varphi\,d\varphi = \frac23\int_0^{\pi/2}\left(\underset{\frac43}{-\frac23+2}\right)\cos\varphi\,d\varphi = \frac89\int_0^{\pi/2}\cos\varphi\,d\varphi =$

$\frac89\sin\varphi\Big|_0^{\pi/2} = \frac89$

j. $\int_0^{2\pi}\int_0^{\sqrt3}\int_0^{3-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt3}3r-r^3\,dr\,d\theta =$



above
$z = 3-r^2$
(paraboloid)

$\int_0^{2\pi}\frac32r^2-\frac14r^4\Big|_0^{\sqrt3}d\theta = \int_0^{2\pi}\frac32(3)-\frac14(9)\,d\theta = \frac94\cdot2\pi = \frac92\pi$

k. $\int_0^{2}\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{16-x^2-y^2}}\sqrt{x^2+y^2}\,dz\,dy\,dx$

top half of hemisphere
first octant only
inside cylinder $x^2+y^2=4$

need 2 integrals in Spherical

$\int_0^{\pi/2}\int_0^{2}\int_0^{\sqrt{16-r^2}} r^2\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{2} r^2\sqrt{16-r^2}\,dr\,d\theta$

$r=4\sin\alpha$
$dr=4\cos\alpha\,d\alpha$

$\int(4\sin\alpha)^2\,4\cos\alpha\cdot4\cos\alpha\,d\alpha$ $\quad\sqrt{16-16\sin^2\alpha}=\sqrt{16(1-\sin^2\alpha)} = 4\cos\alpha$

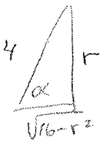
$256\int\sin^2\alpha\cos^2\alpha\,d\alpha = 64\int(1-\cos2\alpha)(1+\cos2\alpha)\,d\alpha = 64\int1-\cos^22\alpha\,d\alpha =$

$64\int1-\frac12(1+\cos4\alpha)\,d\alpha = 64\int\frac12-\frac12\cos4\alpha\,d\alpha = 32\int1-\cos4\alpha\,d\alpha =$

7k) cont'd

$$32\left[\alpha - \frac{1}{4}\sin 4\alpha\right] = 32\left[\alpha - \frac{1}{4}\cdot 2\sin 2\alpha\cos 2\alpha\right] = 32\left[\alpha - \cancel{\frac{1}{4}4}\sin\alpha\cos\alpha\left(1-2\sin^2\alpha\right)\right]$$

$\sin\alpha = \frac{r}{4}$

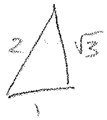


$$32\left[\arcsin\left(\frac{r}{4}\right) - \frac{r}{4}\cdot\frac{\sqrt{16-r^2}}{4}\left(1-\cancel{2}\cdot\frac{r^2}{\underset{8}{\cancel{16}}}\right)\right] = 32\left[\arcsin\left(\frac{r}{4}\right) - \frac{r\sqrt{16-r^2}}{16}\left(\frac{8-r^2}{8}\right)\right]$$

$$= 32\arcsin\left(\frac{r}{4}\right) - \frac{1}{\cancel{4}}\left(\overset{2}{\cancel{8}}r\sqrt{16-r^2}\right) + \frac{1}{4}\left(r^3\sqrt{16-r^2}\right)\Big|_0^2 =$$

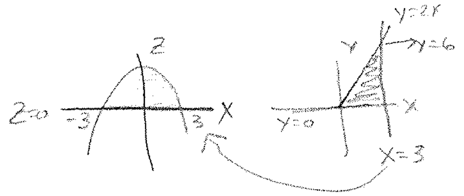
$$32\arcsin\left(\frac{1}{2}\right) - 2(2)(2\sqrt{3}) + \frac{1}{\underset{2}{\cancel{4}}}(8)(2\sqrt{3}) = \overset{16}{\cancel{32}}\left(\frac{\pi}{\underset{3}{\cancel{6}}}\right) - 4\sqrt{3}$$

$-8\sqrt{3}$ $+4\sqrt{3}$



$$\int_0^{\pi/2}\left(\frac{16\pi}{3}-4\sqrt{3}\right)d\theta = \left(\frac{16\pi}{3}-4\sqrt{3}\right)\pi/2 = \frac{8\pi}{3}-2\sqrt{3}$$

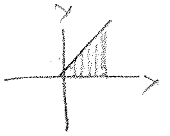
8) a. $z = 9-x^2$ $y=2x$ $x=0,\ y=0,\ z=0$

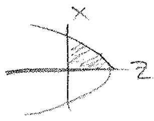


$$\int_0^3\int_0^{2x}\int_0^{9-x^2} dz\,dy\,dx = \int_0^6\int_{y/2}^3\int_0^{9-x^2} dz\,dx\,dy =$$

$x = y/2$

$$\int_0^3\int_0^{9-x^2}\int_0^{2x} dy\,dz\,dx = \int_0^9\int_0^{\sqrt{9-z}}\int_0^{2x} dy\,dx\,dz =$$

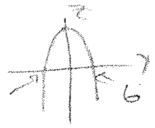




$x = \sqrt{9-z}$

$$\int_0^9\int_0^{2\sqrt{9-z}}\int_0^{y/2} dx\,dy\,dz = \int_0^6\int_0^{y/2}\int_0^{2\sqrt{9-z}} dx\,dz\,dy$$

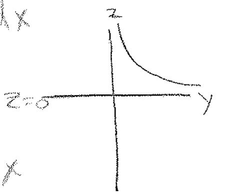
$z = 9 - \frac{y^2}{4}$

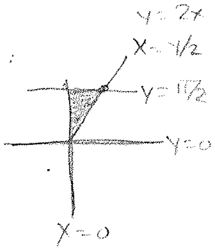


$4z = 36-y^2$

$y = \sqrt{36-4z} = 2\sqrt{9-z}$

b) $$\int_0^{\pi/2}\int_0^{y/2}\int_0^{1/y}\sin y\,dz\,dx\,dy = \int_0^{\pi}\int_0^{2x}\int_0^{1/4}\sin y\,dz\,dy\,dx$$





$$\int_0^{2/\pi}\int_0^{1/2z}\int_0^{1/z}\sin y\,dy\,dx\,dz = \int_0^{\pi}\int_0^{1/2x}\int_0^{1/z}\sin y\,dy\,dz\,dx$$

$y = 2(\pi/2) = \pi$

$$\int_0^{2/\pi}\int_0^{1/z}\int_0^{y/2}\sin y\,dx\,dy\,dz = \int_0^{\pi/2}\int_0^{1/y}\int_0^{y/2}\sin y\,dx\,dz\,dy$$

$z = \frac{1}{y} \Rightarrow y = \frac{1}{z}$

$z = \frac{2}{\pi}$ $2x = \frac{1}{z}$ $x = \frac{1}{2z}$