Math 2153 Homework #7 Key

1) a. $\frac{1}{1}\int_0^1\int_0^1\int_0^1 z^2+4\,dz\,dy\,dx$ $\quad V=$  $=1$

$\int_0^1\int_0^1 \frac{1}{3}z^3+4z\Big|_0^1 dy\,dx = \int_0^1\int_0^1 \frac{13}{3}\,dy\,dx = \int_0^1 \frac{13}{3}\,dx = \frac{13}{3}$

b. $\frac{3}{8\sqrt{2}\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\sqrt{2}} (\rho\sin\varphi\cos\theta+\rho\sin\varphi\sin\theta)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$ $\quad V=\frac{4}{3}\pi(\sqrt{2})^3$

$\frac{3}{8\sqrt{2}\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\sqrt{2}} \rho^3(\sin^2\varphi)(\cos\theta+\sin\theta)\,d\rho\,d\varphi\,d\theta =$

$\frac{3}{8\sqrt{2}\pi}\int_0^{2\pi}\int_0^{\pi}\frac{1}{4}\rho^4\Big|_0^{\sqrt{2}}\sin^2\varphi(\cos\theta+\sin\theta)\,d\varphi\,d\theta = \frac{3}{8\sqrt{2}\pi}\int_0^{2\pi}\int_0^{\pi}\frac{1}{2}(1-\cos 2\varphi)\,d\varphi\,d\theta$

$\frac{3}{16\sqrt{2}\pi}\int_0^{2\pi}\varphi-\frac{1}{2}\sin 2\varphi\Big|_0^{\pi}d\theta = \frac{3}{16\sqrt{2}\cancel{\pi}}\int_0^{2\pi}\cancel{\pi}\,d\theta = \frac{3}{\underset{8}{\cancel{16}}\sqrt{2}}\cdot\cancel{2}\pi = \frac{3\pi}{8\sqrt{2}}$

c. $\int_0^3\int_0^x\int_0^{9-x^2} dz\,dy\,dx = \text{Volume} = \int_0^3\int_0^x 9-x^2\,dy\,dx = \int_0^3 x(9-x^2)\,dx = \int_0^3 9x-x^3\,dx$

$= \frac{9}{2}x^2-\frac{1}{4}x^4\Big|_0^3 = \frac{81}{2}-\frac{81}{4} = \frac{81}{4}$

$\bar{f} = \frac{4}{81}\int_0^3\int_0^x\int_0^{9-x^2} 3x+2y-z+10\,dz\,dy\,dx = \frac{4}{81}\int_0^3\int_0^x (9-x^2)(3x+2y+10)-\frac{1}{2}(9-x^2)^2\,dy\,dx$

$81-18x^2+x^4$

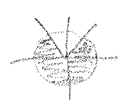
$\frac{4}{81}\int_0^3\int_0^x 27x-3x^3+18y-2x^2y+90-10x^2-\frac{81}{2}+9x^2-\frac{1}{2}x^4\,dy\,dx$

$\frac{4}{81}\int_0^3\int_0^x 27x-3x^3+18y-2x^2y+\frac{99}{2}-x^2-\frac{1}{2}x^4\,dy\,dx =$

$\frac{4}{81}\int_0^3 27x^2-3x^4+9x^2-x^4+\frac{99}{2}x-x^3-\frac{1}{2}x^5\,dx = \frac{4}{81}\int_0^3 36x^2-4x^4+\frac{99}{2}x-x^3-\frac{1}{2}x^5\,dx$

$\frac{4}{81}\left[12x^3-\frac{4}{5}x^5+\frac{99}{4}x^2-\frac{1}{4}x^4-\frac{1}{12}x^6\right]_0^3 = \frac{4}{81}\left[324-\frac{972}{5}+\frac{891}{4}-\frac{81}{4}-\frac{243}{4}\right] =$

$\cancel{\frac{4}{81}}\left[\frac{\overset{67}{\cancel{5427}}}{\underset{5}{\cancel{20}}}\right] = \frac{67}{5}$

2) a. $\int_0^{2\pi}\int_{\pi/4}^{\pi}\int_0^4 \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{1}{3}\int_0^{2\pi}\int_{\pi/4}^{\pi}\rho^3\Big|_0^4\sin\varphi\,d\varphi\,d\theta =$ 

$\frac{64}{3}\int_0^{2\pi}\int_{\pi/4}^{\pi}\sin\varphi\,d\varphi\,d\theta = \frac{64}{3}\int_0^{2\pi}-\cos\varphi\Big|_{\pi/4}^{\pi}d\theta = \frac{64}{3}\int_0^{2\pi}\left[-(-1)+\frac{\sqrt{2}}{2}\right]d\theta = \frac{32(2+\sqrt{2})}{3}\cdot 2\pi$

$= \frac{64\pi(2+\sqrt{2})}{3}$

2 e) cont'd $-\underline{\pi a^4}\,[0-1]\int_0^{\pi/2} d\theta = \pi a^4 \cdot \frac{\pi}{2} = \frac{\pi^2 a^4}{2}$ ← This is the correct answer according to wikipedia! :)

3) a. $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{v} \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

$x = au+bv$
$y = cu+dv$

b. $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$ rotation

$x = u\cos\theta - v\sin\theta$
$y = u\sin\theta + v\cos\theta$

c. $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} e^u\sin v & e^u\cos v \\ e^u\cos v & -e^u\sin v \end{vmatrix} = -e^{2u}\sin^2 v - e^{2u}\cos^2 v = -e^{2u}$

$x = e^u\sin v$
$y = e^u\cos v$

d. $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 1 & 1 \end{vmatrix} = \frac{1}{v} + \frac{u}{v^2} = \frac{v+u}{v^2}$

$x = \frac{u}{v}$
$y = u+v$

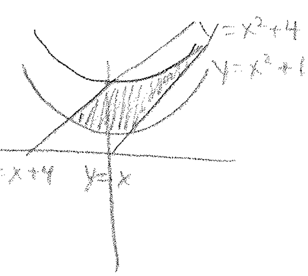
e. $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \pm\frac{1}{2}(\frac{1}{4}+u+v)^{-1/2} & \pm\frac{1}{2}(\frac{1}{4}+u+v)^{-1/2} \\ \pm\frac{1}{2}(\frac{1}{4}+u+v)^{-1/2} & 1\pm\frac{1}{2}(\frac{1}{4}+u+v)^{-1/2} \end{vmatrix} = \frac{1}{2}(\frac{1}{4}+u+v)^{-1/2} \pm \cancel{\frac{1}{4}(\frac{1}{4}+u+v)^{-1}} \mp \cancel{\frac{1}{4}(\frac{1}{4}+u+v)^{-1}} =$

$\frac{1}{2\sqrt{\frac{1}{4}+u+v}}$

$u = x^2-y$
$v = y-x$
$y = v+x$

$$\left.\begin{aligned} u &= x^2 - v - x \\ \tfrac{1}{4}+u+v &= x^2 - x + \tfrac{1}{4} \\ \sqrt{(\tfrac{1}{4}+u+v)} &= (x-\tfrac{1}{2})^{\cancel{2}} \\ x &= \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4}+u+v} \\ y &= v + \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4}+u+v} \end{aligned}\right\}$$ Solve for, $x(u,v)$, $y(u,v)$ first!

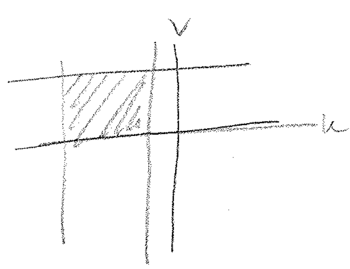
4) a.



u
$x^2-y=-4$
$x^2-y=-1$
$y-x=0$
$y-x=4$

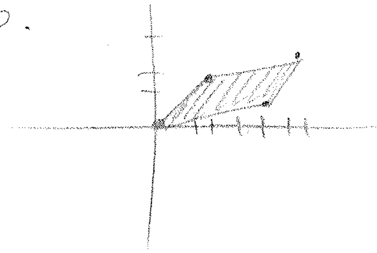
$-4 \le u \le -1$

$0 \le v \le 4$



this is exactly like 3e.

4)b.



$(0,0) \to (2,2)$

$m = \frac{2-0}{2-0} = \frac{2}{2} = 1$

$y = x$

$(0,0) \to (4,1)$

$m = \frac{1-0}{4-0} = \frac{1}{4}$

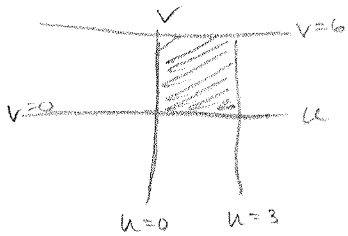
$y = \frac{1}{4}x$

$4y = x$

$x - y = 0$
$x - y = 3$

$u = x - y$

$0 \le u \le 3$



$(2,2) \to (6,3)$

$m = \frac{3-2}{6-2} = \frac{1}{4}$

$y - 2 = \frac{1}{4}(x-2)$

$y - 2 = \frac{1}{4}x - \frac{1}{2}$

$y = \frac{1}{4}x + \frac{3}{2}$

$4y = x + 6$

$(4,1) \to (6,3)$

$m = \frac{3-1}{6-4} = \frac{2}{2} = 1$

$y - 1 = x - 4$

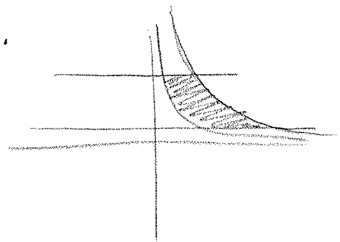
$y = x - 3$

$4y - x = 0$
$4y - x = 6$

$4y - x = v$

$0 \le v \le 6$

c.



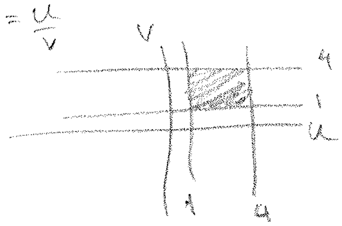
$y = \frac{1}{x}$ $\quad y = \frac{4}{x}$

$y = 1, \; y = 4$

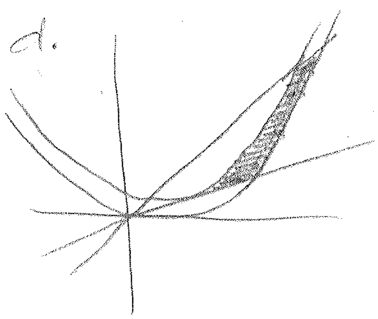
$1 \le u \le 4$ $\qquad 1 \le v \le 4$

$u = xy$ $\quad x = \frac{u}{v}$

$y = v$



d.



$y = 2x, \; y = 3x$

$\frac{y}{x} = 2 \quad \frac{y}{x} = 3$

$u = \frac{y}{x}$

$2 \le u \le 3$

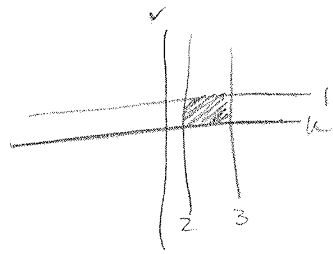
$y = x^2 \quad y = x^2 + 1$

$y - x^2 = 0$

$y - x^2 = 1$

$v = y - x^2$

$0 \le v \le 1$



5)a. $f(x,y) = \sin 3x \cos 4y$

$\nabla f = 3\cos 3x \cos 4y \, \hat{i} - 4 \sin 3x \sin 4y \, \hat{j}$

b. $f(x,y,z) = \frac{y}{z} + \frac{z}{x} - \frac{xz}{y}$

$\nabla f = \left(-\frac{z}{x^2} - \frac{z}{y}\right)\hat{i} + \left(\frac{1}{z} + \frac{xz}{y^2}\right)\hat{j} + \left(-\frac{y}{z^2} - \frac{x}{y}\right)\hat{k}$

c. $f(x,y,z) = x \arcsin(yz)$

$\nabla f = \arcsin(yz)\,\hat{i} + \frac{xz}{\sqrt{1-y^2z^2}}\,\hat{j} + \frac{xy}{\sqrt{1-y^2z^2}}\,\hat{k}$

6) d. $\vec{F}(x,y) = \frac{x}{\sqrt{x^2+y^2}}\hat{\imath} + \frac{y}{\sqrt{x^2+y^2}}\hat{\jmath}$ $\quad f(x,y) = \sqrt{x^2+y^2} + K$ potential function

$\int \frac{x}{\sqrt{x^2+y^2}}\,dx \quad u = x^2+y^2, \ \frac{1}{2}du = x\,dx$ $\qquad \int \frac{y}{\sqrt{x^2+y^2}}\,dy \quad u = x^2+y^2, \ \frac{1}{2}du = y\,dy \Rightarrow \sqrt{x^2+y^2} + C(x)$

$\int \frac{1}{2}u^{-1/2}\,du$

$\frac{1}{2}2u^{1/2} \Rightarrow \sqrt{x^2+y^2} + C(y)$

$\frac{d}{dy}\left[x(x^2+y^2)^{-1/2}\right] = x\left(-\frac{1}{2}\right)(x^2+y^2)^{-3/2}(2y) = \frac{-xy}{(x^2+y^2)^{3/2}}$ ✓ $\quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

$\frac{d}{dx}\left[y(x^2+y^2)^{-1/2}\right] = y\left(-\frac{1}{2}\right)(x^2+y^2)^{-3/2}(2x) = \frac{-xy}{(x^2+y^2)^{3/2}}$ ✓ $\quad$ Conservative

e. $\vec{F}(x,y) = \frac{1}{x}\hat{\imath} - \frac{1}{y}\hat{\jmath}$ $\qquad \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y}$ conservative

$\int \frac{1}{x}\,dx = \ln x + C(y)$

$-\int \frac{1}{y}\,dy = -\ln y + C(x)$ $\qquad f(x,y) = \ln x - \ln y + K = \ln(x/y) + K$

f. $\vec{F}(x,y) = 2xye^{x^2y}\hat{\imath} + x^2e^{x^2y}\hat{\jmath}$ $\qquad \frac{\partial N}{\partial x} = 2xe^{x^2y} + x^2 \cdot 2xye^{x^2y}$ ✓

$\int 2xye^{x^2y}\,dx \quad u = x^2y, \ du = 2xy\,dx$ $\qquad \frac{\partial M}{\partial y} = 2xe^{x^2y} + 2xye^{x^2y}\cdot x^2$ conservative

$= e^{x^2y} + C(y)$

$\int x^2e^{x^2y}\,dy = e^{x^2y} + C(x)$ $\qquad f(x,y) = e^{x^2y} + K$

g. $\vec{F}(x,y) = 3x^2y^2\hat{\imath} + 3x^3y\hat{\jmath}$ $\qquad \frac{\partial N}{\partial x} = 9x^2y \neq \frac{\partial M}{\partial y} = 6x^2y$

not conservative

h. $\vec{F}(x,y,z) = \sin y\,\hat{\imath} - x\cos y\,\hat{\jmath} + \hat{k}$

$$\vec{\nabla}\times\vec{F}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin y & -x\cos y & 1 \end{vmatrix} = (0-0)\hat{\imath} - (0-0)\hat{\jmath} + (-\cos y - \cos y)\hat{\jmath}$$

$= -2\cos y\,\hat{k} \neq \vec{0}$ not conservative

i. $\vec{F}(x,y,z) = y^2z^3\hat{\imath} + 2xyz^3\hat{\jmath} + 3xy^2z^3\hat{k}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^3 \end{vmatrix} = (6xyz^3 - 6xyz^2)\hat{\imath} - (3y^2z^3 - 3y^2z^2)\hat{\jmath} + (2yz^3 - 3y^2z^2)\hat{k}$$

$\neq \vec{0}$ not conservative

7) a. $\vec{F}\times\vec{G} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ xy & -2y & x^2\sin z \\ \ln x & 2e^z & -3y \end{vmatrix} = (6y^2 - 2x^2e^z\sin z)\hat{\imath} - (-3xy^2 - x^2\ln x\sin z)\hat{\jmath} + (2xye^z + 2y\ln x)\hat{k}$

$$\vec{\nabla}\times(\vec{F}\times\vec{G}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6y^2-2x^2e^z\sin z & 3xy^2+x^2\ln x\sin z & 2xye^z+2y\ln x \end{vmatrix} =$$

$$(2xe^z + 2\ln x - x^2\ln x\cos z)\hat{\imath} - (2ye^z + \frac{2y}{x} + 2x^2e^z\sin z + 2x^2e^z\cos z)\hat{\jmath}$$
$$+ (3y^2 + 2x\ln x\sin z + x\sin z - 12y)\hat{k}$$

b. $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & -2y & x^2\sin z \end{vmatrix} = (0-0)\hat{\imath} - (2x\sin z - 0)\hat{\jmath} + (0-x)\hat{k}$

$\vec{\nabla}\times(\nabla\times F) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & -2x\sin z & -x \end{vmatrix} = (0 + 2x\cos z)\hat{\imath} - (-1-0)\hat{\jmath} + (-2\sin z - 0)\hat{k}$

c. See a) for $F\times G$

$\vec{\nabla}\cdot(\vec{F}\times\vec{G}) = 4xe^z\sin z + 6xy + 2xye^z$

d. $\vec{\nabla}\cdot(f\vec{F}) = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle \cdot [\langle (xy+z^2-6)xy, (xy+z^2-6)(-2y), (xy+z^2-6)(x^2\sin z)\rangle]$

$= (y)(xy) + (xy+z^2-6)(y) + (x)(-2y) + (xy+z^2-6)(-2) + (2z)(x^2\sin z) + (xy+z^2-6)(x^2\cos z)$

$xy^2 + xy^2 + yz^2 - 6y - 2xy - 2xy - 2z^2 + 12 + 2x^2z\sin z + x^3y\cos z + x^2z^2\cos z - 6x^2\cos z$

$2xy^2 + yz^3 - 6y - 4xy - 2z^2 + 12 + 2x^2z\sin z + x^3y\cos z + x^2z^2\cos z - 6x^2\cos z$

e. $\vec{\nabla}f = y\hat{\imath} + x\hat{\jmath} + 2z\hat{k}$ $\quad\quad \vec{\nabla}\cdot\vec{\nabla}f = \nabla^2 f$ (btw)

$\vec{\nabla}\cdot\vec{\nabla}f = 0 + 0 + 2 = 2$

8) a. $x^2+y^2=9$

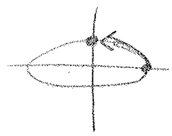
$x=3\cos t$

$y=3\sin t \Rightarrow r_1(t)=3\cos t\,\hat{\imath}+3\sin t\,\hat{\jmath}$ $\quad 0\le t\le 2\pi$

Or $x=3\cos 2t$

$y=3\sin 2t \Rightarrow r_2(t)=3\cos 2t\,\hat{\imath}+3\sin 2t\,\hat{\jmath}$ $\quad 0\le t\le \pi$

b) $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$



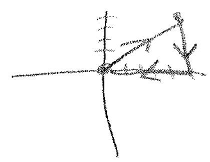
$x=4\cos t$

$y=3\sin t \Rightarrow r_1(t)=4\cos t\,\hat{\imath}+3\sin t\,\hat{\jmath}$ $\quad 0\le t\le \pi/2$

Or $x=4\cos 3t$

$y=3\sin 3t \Rightarrow r_2(t)=4\cos 3t\,\hat{\imath}+3\sin 3t\,\hat{\jmath}$ $\quad 0\le t\le \pi/6$

c) $(0,0)\to(5,4)\to(5,0)\to(0,0)$



$r_{11}=5t\,\hat{\imath}+4t\,\hat{\jmath}$ $\quad 0\le t\le 1$ (1st leg)

$r_{12}=5\,\hat{\imath}+(-4(t-1)+4)\,\hat{\jmath}$ $\quad 1\le t\le 2$ (2nd leg)

$r_{13}=(-5(t-2)+5)\,\hat{\imath}+0\,\hat{\jmath}$ $\quad 2\le t\le 3$ (3rd leg)

Or

$r_{21}=10t\,\hat{\imath}+8t\,\hat{\jmath}$ $\quad 0\le t\le \frac{1}{2}$ (1st leg)

$r_{22}=5\,\hat{\imath}+(-8(t-1/2)+4)\,\hat{\jmath}$ $\quad \frac{1}{2}\le t\le 1$

$r_{23}=(-5(t-1)+5)\,\hat{\imath}+0\,\hat{\jmath}$ $\quad 1\le t\le 2$

d. $(0,0,0)\to(1,0,0)\to(1,0,1)\to(1,1,1)$

$r_{11}=t\,\hat{\imath}+0\,\hat{\jmath}+0\,\hat{k}$ $\quad 0\le t\le 1$ 1st leg.

$r_{12}=1\,\hat{\imath}+0\,\hat{\jmath}+(t-1)\,\hat{k}$ $\quad 1\le t\le 2$ 2nd leg.

$r_{13}=1\,\hat{\imath}+(t-2)\,\hat{\jmath}+1\,\hat{k}$ $\quad 2\le t\le 3$ 3rd leg.

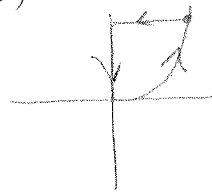
Or

$r_{21}=\sin t\,\hat{\imath}+0\,\hat{\jmath}+0\,\hat{k}$ $\quad 0\le t\le \pi/2$ 1st leg

$r_{22}=1\,\hat{\imath}+0\,\hat{\jmath}-\cos t\,\hat{k}$ $\quad \frac{\pi}{2}\le t\le \pi$ 2nd leg.

$r_{23}=1\,\hat{\imath}-\sin t\,\hat{\jmath}+1\,\hat{k}$ $\quad \pi\le t\le \frac{3\pi}{2}$ 3rd leg.

8) e. $y = x^2$ $(0,0) \to (2,4)$ then $(2,4) \to (0,4) \to (0,0)$



$r_{11} = t\hat{\imath} + t^2\hat{\imath}$ $\quad 0 \le t \le 2$

$r_{12} = \left(-2(t-2)+2\right)\hat{\imath} + 4\hat{\jmath}$ $\quad 2 \le t \le 3$

$r_{13} = 0\hat{\imath} + \left(-4(t-3)+4\right)\hat{\jmath}$ $\quad 3 \le t \le 4$

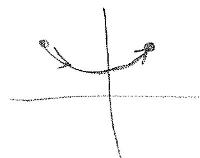
or

$r_{21} = 2\sin t\,\hat{\imath} + 4\sin^2 t\,\hat{\jmath}$ $\quad 0 \le t \le 1$

$r_{22} = \left(-1(t-1)+2\right)\hat{\imath} + 4\hat{\jmath}$ $\quad 1 \le t \le 3$

$r_{23} = 0\hat{\imath} + \left(-8(t-3)+4\right)\hat{\jmath}$ $\quad 3 \le t \le \frac{7}{2}$

f. $\frac{y^2}{16} - \frac{x^2}{4} = 1$ $\quad (-3, 2\sqrt{13}) \to (3, 2\sqrt{13})$



$r_1 = 4\sinh t\,\hat{\imath} - 2\cosh t\,\hat{\jmath}$

For hyperbolas use cosh/sinh as you would cos/sin for ellipses

$4\sinh t = 3$

$\sinh t = \frac{3}{4}$

$e^t - e^{-t} = \frac{3}{2}$ $\quad (\times e^t)$

$e^{2t} - \frac{3}{2}e^t - 1 = 0$

let $u = e^t$

$u^2 - \frac{3}{2}u - 1 = 0$

$2u^2 - 3u - 2 = 0$

$(2u+1)(u-2) = 0$

$\cancel{u = -\tfrac{1}{2}}$ $\quad u = 2$

$e^t > 0$ $\quad t = \ln 2$

$4\sinh t = -3$

$\sinh t = -\frac{3}{4}$

$e^{2t} + \frac{3}{2}e^t - 1 = 0$

let $u = e^t$

$u^2 + \frac{3}{2}u - 1 = 0$

$2u^2 + 3u - 2 = 0$

$(2u-1)(u+2) = 0$

$2u = 1$ $\quad \cancel{u = -2}$

$u = \frac{1}{2}$ $\quad e^t > 0$

$t = \ln\frac{1}{2} = -\ln 2$

$-\ln 2 \le t \le \ln 2$

$r_2 = 4\sinh 2t\,\hat{\imath} - 2\cosh 2t\,\hat{\jmath}$ $\qquad -\ln\sqrt{2} \le t \le \ln\sqrt{2}$

$u = e^{2t}$

$2t = \ln 2$

$t = \frac{1}{2}\ln 2 = \ln\sqrt{2}$

similarly

$2t = \ln\frac{1}{2}$

$t = \ln\frac{1}{\sqrt{2}} = -\ln\sqrt{2}$