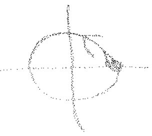
Math 2153 Homework #8 Key

1) a $x^2+y^2=9$ starting at (3,0) counterclockwise

$\vec{r}(t) = \underbrace{3\cos t}_{x}\,\hat{\imath} + \underbrace{3\sin t}_{y}\,\hat{\jmath} \qquad 0 \le t \le 2\pi$

$\vec{r}\,'(t) = \underbrace{-3\sin t}_{dx}\,\hat{\imath} + \underbrace{3\cos t}_{dy}\,\hat{\jmath}$



$A = \frac{1}{2}\int_0^{2\pi} \underbrace{3\cos t}_{x}\underbrace{(3\cos t)}_{dy} - \underbrace{3\sin t}_{y}\underbrace{(-3\sin t)}_{dx}\,dt$

$= \frac{1}{2}\int_0^{2\pi} 9\cos^2 t + 9\sin^2 t\,dt = \frac{1}{2}\int_0^{2\pi} 9\,dt = \frac{9\cdot 2\pi}{2} = 9\pi$

(circle of radius 3 $\Rightarrow A = \pi(3)^2 = 9\pi$)

b) $(0,0) \to (5,4) \to (5,0) \to (0,0)$

$\vec{r}_1(t) = 5t\,\hat{\imath} + 4t\,\hat{\jmath} \quad 0 \le t \le 1 \qquad \vec{r}_1\,'(t) = 5\hat{\imath} + 4\hat{\jmath}$

$\vec{r}_2(t) = 5\hat{\imath} - 4(t)\hat{\jmath} \quad 0 \le t \le 1 \qquad \vec{r}_2\,'(t) = 0\hat{\imath} - 4\hat{\jmath}$

$\vec{r}_3(t) \quad -5t\,\hat{\imath} + 0\hat{\jmath} \quad 0 \le t \le 1 \qquad \vec{r}_3\,'(t) = -5\hat{\imath} + 0\hat{\jmath}$

$A = \frac{1}{2}\left[\int_{C_1} x\,dy - y\,dx + \int_{C_2} x\,dy - y\,dx + \int_{C_3} x\,dy - y\,dx\right] =$

$\frac{1}{2}\left[\int_0^1 5t\cdot 4 - 4t(5)\,dt + \int_0^1 5(-4) - (-4)(0)\,dt + \int_0^1 \cancel{(-5t)(0) - (0)(-5)}\,dt\right]$

$\frac{1}{2}\left[\int_0^1 \cancel{20t - 20t}^{\,0}\,dt + \int_0^1 -20\,dt\right] = \frac{1}{2}\left[\int_0^1 -20\,dt\right] = \frac{1}{2}\left[-20t\Big|_0^1\right] =$

$\frac{1}{2}[-20] = -10 \qquad A = \frac{1}{2}bh = \frac{1}{2}(5)(4) = 10$

the reason the area comes out negative here is because we are tracing the curve in the clockwise rather than counterclockwise direction

c) along $y = x^2$ from $(0,0) \to (2,4)$ then $(2,4) \Rightarrow (0,4) \to (0,0)$

$\vec{r}_1(t) = t\,\hat{\imath} + t^2\,\hat{\jmath} \quad 0 \le t \le 2 \qquad \vec{r}_1\,'(t) = 1\hat{\imath} + 2t\,\hat{\jmath}$

$\vec{r}_2(t) = -2t\,\hat{\imath} + 4\hat{\jmath} \quad 0 \le t \le 1 \qquad \vec{r}_2\,'(t) = -2\hat{\imath} + 0\hat{\jmath}$

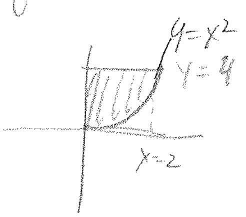
$\vec{r}_3(t) = 0\hat{\imath} - 4t\,\hat{\jmath} \quad 0 \le t \le 1 \qquad \vec{r}_3\,'(t) = 0\hat{\imath} - 4\hat{\jmath}$

$\frac{1}{2}\left[\int_0^2 t(2t) - t^2(1)\,dt + \int_0^1 (-2t)(0) - (4)(-2)\,dt + \int_0^1 (0)(-4) - \cancel{(-4t)(0)}^{\,0}\,dt\right] =$

1)e continued

$$\frac{1}{2}\left[\int_0^2 t^2\,dt + \int_0^1 8\,dt\right] = \frac{1}{2}\left[\frac{1}{3}t^3\Big|_0^2 + 8t\Big|_0^1\right] = \frac{1}{2}\left[\frac{8}{3}+8\right] = \frac{1}{2}\left[\frac{32}{3}\right] = \frac{16}{3}$$

you can check this area against the usual integration method



$$\int_0^2\int_{x^2}^4 dy\,dx = \int_0^2 4-x^2\,dx = 4x-\frac{1}{3}x^3\Big|_0^2 = 8-\frac{8}{3} = \frac{16}{3}$$

2) a. $8\int_C (xy)\,ds$

$\vec{r}\,'(t) = -2\sin t\,\hat{i} + 2\cos t\,\hat{j} + 1\,\hat{k}$

$ds = \|\vec{r}\,'(t)\|\,dt = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{4+1} = \sqrt{5}$

1/4 of a turn

Since each quarter will be equal due to symmetry of the helix & the mass function in each octant (in particular, mass doesn't depend on z here)

$$8\int_0^{\pi/2}(2\cos t)(2\sin t)\sqrt{5}\,dt =$$

$$32\sqrt{5}\int_0^{\pi/2}\underset{du}{\cos t}\,\underset{u}{\sin t}\,dt = \frac{\overset{16}{\cancel{32}}\sqrt{5}}{\cancel{2}}\sin^2 t\Big|_0^{\pi/2} =$$

$16\sqrt{5}(1)$

b. $\vec{r}\,'(t) = -3\sin t\,\hat{i} + 2\hat{j} + \cos t\,\hat{j}$ $\quad 0 \le t \le \pi$ $\quad \rho = kz$

$\int_0^{\pi} k(\sin t)\,ds$

$ds = \|r'(t)\|\,dt = \sqrt{9\sin^2 t + 4 + \cos^2 t}\,dt$

$\sqrt{9\sin^2 t + 9\cos^2 t + 4 - 8\cos^2 t}\,dt$

$= \sqrt{9+4-8\cos^2 t} = \sqrt{13-8\cos^2 t}$

$k\int_0^{\pi}\sin t\sqrt{13-8\cos^2 t}\,dt$

$u = \cos t$

$-du = +\sin t\,dt$

$\cos(0) = 1$

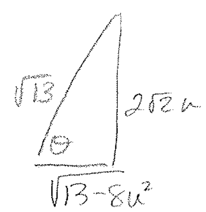
$\cos(\pi) = -1$

$$\Rightarrow -k\int_1^{-1}\sqrt{13-8u^2}\,du = 0$$

$$= k\int_{-1}^{1}\sqrt{13-8u^2}\,du$$

$2\sqrt{2}u = \sqrt{13}\sin\theta$

$\cancel{2\sqrt{2}}\,du = \frac{\sqrt{13}}{2\sqrt{2}}\cos\theta\,d\theta$



$$k\int_{\sin^{-1}\left(\frac{-2\sqrt{2}}{\sqrt{13}}\right)}^{\sin^{-1}\left(\frac{2\sqrt{2}}{\sqrt{13}}\right)} \frac{\sqrt{13}}{2\sqrt{2}}\cos\theta\cdot\sqrt{13}\cos\theta\,d\theta$$

$\sqrt{13-13\sin^2\theta} = \sqrt{13}\cos\theta$

$\frac{2\sqrt{2}(1)}{\sqrt{13}} = \cancel{\sqrt{13}}\sin\theta$

$\theta = \sin^{-1}\left(\frac{2\sqrt{2}}{\sqrt{13}}\right)$

$$\frac{13k}{2\sqrt{2}}\int_{\sin^{-1}\left(\frac{-2\sqrt{2}}{\sqrt{13}}\right)}^{\sin^{-1}\left(\frac{2\sqrt{2}}{\sqrt{13}}\right)}\cos^2\theta\,d\theta =$$

$$\frac{13k}{4\sqrt{2}}\int_{\sin^{-1}\left(-\frac{2\sqrt{2}}{\sqrt{13}}\right)}^{\sin^{-1}\left(\frac{2\sqrt{2}}{\sqrt{13}}\right)} 1+\cos 2\theta\,d\theta = \frac{13k}{4\sqrt{2}}\left[\theta + \frac{1}{2}\overset{\sin\theta\cos\theta}{\sin 2\theta}\right]_{\sin^{-1}\left(-\frac{2\sqrt{2}}{\sqrt{13}}\right)}^{\sin^{-1}\left(\frac{2\sqrt{2}}{\sqrt{13}}\right)}$$

2b continued

when $u=1$

$$\frac{13k}{4\sqrt{2}}\left[\sin^{-1}\left(\frac{2\sqrt{2}}{\sqrt{13}}\right)+\sin^{-1}\left(\frac{+2\sqrt{2}}{\sqrt{13}}\right)+\frac{2\sqrt{2}}{\sqrt{13}}\left(\frac{\sqrt{5}}{\sqrt{13}}\right)+\frac{+2\sqrt{2}}{\sqrt{13}}\left(\frac{\sqrt{5}}{\sqrt{13}}\right)\right]$$

$$\frac{13k}{2\sqrt{2}}\left[\arcsin\left(\frac{2\sqrt{2}}{\sqrt{13}}\right)+\frac{2\sqrt{10}}{13}\right]=6.381\ldots(k)$$

which is what you get doing the integral numerically

c. $\int_C \sin(t)\,ds \qquad \vec{r}\,'(t)=(2-4\cos t)\hat{\imath}+(4\sin t)\hat{\jmath} \qquad 0\le t\le\pi$

$ds=\|\vec{r}\,'(t)\|\,dt=\sqrt{4-16\cos t+16\cos^2 t+16\sin^2 t}=$

$\sqrt{4+16-16\cos t}=\sqrt{20-16\cos t}=2\sqrt{5-4\cos t}\,dt$

$2\int_0^\pi \sin t\sqrt{5-4\cos t}\,dt \qquad 5-4\cos t=u$

$\cancel{+4}\sin t\,dt=\frac{du}{4}$

$$\frac{2}{4}\int u^{1/2}\,du=\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}u^{3/2}\Rightarrow\frac{1}{3}(5-4\cos t)^{3/2}\Big|_0^\pi=\frac{1}{3}\left[\left(\sqrt{5-4(-1)}\right)^3-\left(\sqrt{5-4(1)}\right)^3\right]=$$

$$\frac{1}{3}\left[9^{3/2}-1^{3/2}\right]=\frac{1}{3}[27-1]=\frac{26}{3}$$

d. $\vec{r}\,'(t)=2t\hat{\imath}+4\sin t\cos t\,\hat{\jmath}-2\csc^2 t \qquad \rho(x,y,z)=kyz \qquad \pi/4\le t\le\pi/2$

$\int_C k(2\sin^2 t)\cdot(2\cot(t))\,ds \qquad ds=\|\vec{r}\,'(t)\|=$

$\sin^2 t\cdot\frac{\cos t}{\sin t}$

$\sqrt{4t^2+4\sin^2\cos^2 t+4\csc^4 t}= \qquad \frac{1}{\sin^4 t}$

$$k\int_{\pi/4}^{\pi/2}4\,\cancel{\sin^2 t}\,\frac{\cos t}{\sin t}\cdot2\,\frac{\sqrt{t^2\sin^4 t+\sin^6 t\cos^2 t+1}}{\cancel{\sin^2 t}}\,dt \qquad 2\,\frac{\sqrt{t^2\sin^4 t+\sin^6 t\cos^2 t+1}}{\sin^2 t}\,dt$$

$$8k\int_{\pi/4}^{\pi/2}\frac{\cos t}{\sin t}\sqrt{t^2\sin^4 t+\sin^6 t(1-\sin^2 t)+1}\,dt=$$

$u=\sin t \qquad t=\arcsin u$

$du=\cos t\,dt$

$$8k\int_{\pi/4}^{\pi/2}\frac{\cos t}{\sin t}\sqrt{t^2\sin^4 t+\sin^6 t-\sin^8 t+1}\,dt$$

$\sin(\pi/4)=\frac{1}{\sqrt{2}} \qquad \sin(\pi/2)=1$

$$8k\int_{1/\sqrt{2}}^{1}\frac{1}{u}\sqrt{(\arcsin u)^2u^4+u^6-u^8+1}\,du$$

This will have to be done numerically but substitution w/ make it a bit easier to plug into calculator

$\approx 3.649\ldots(k)$

3) answers may vary

a. $x+y-z=6 \Rightarrow z=6-x-y$ let $x=u,\ y=v$

$\vec{r}(u,v)= u\hat{i}+v\hat{j}+(6-x-y)\hat{k}$

b. $\dfrac{4x^2+y^2=16}{16} \Rightarrow \dfrac{x^2}{4}+\dfrac{y^2}{16}=1$ in 2D, this is an ellipse
z is free

$\vec{r}(u,v)= 2\cos u\,\hat{i}+4\sin u\,\hat{j}+v\hat{k}$ $0\le u\le 2\pi$

c. plane at $z=4$ (k coordinate fixed)
all points inside circle $x^2+y^2=9$ (not just $r=3$)

$\vec{r}(u,v)= v\cos u\,\hat{i}+v\sin u\,\hat{j}+4\hat{k}$ $0\le v\le 3$
$0\le u\le 2\pi$

d. surface $x=\sin z$ $0\le z\le\pi$ around z-axis

$x^2+y^2=[\sin z]^2$ let $z=v$ $x=r\cos u$
$r=\sin z$ $y=r\sin u$

$\vec{r}(u,v)= \sin v\cos u\,\hat{i}+\sin v\sin u\,\hat{j}+v\hat{k}$

e. ellipsoid $\dfrac{x^2}{9}+\dfrac{y^2}{4}+\dfrac{z^2}{1}=1$ think spherical analogy w/ ellipse
w/ ρ changing

$\vec{r}(u,v)= 3\sin u\cos v\,\hat{i}+2\sin u\sin v\,\hat{j}$ $\approx \varphi=u,\ \theta=v$
$+\cos u\,\hat{k}$

f. $z=y$ let $y=v,\ x=u$ (free)

$\vec{r}(u,v)= u\hat{i}+v\hat{j}+v\hat{k}$

g. $y=x^{3/2}$ revolved around x-axis

$y^2+z^2=[x^{3/2}]^2=x^3$ treat x^3 like r let $x=v$
y, z like Polar

$\vec{r}(u,v)= v\hat{i}+v^3\cos u\,\hat{j}+v^3\sin u\,\hat{k}$ $y=r\cos u$, $z=r\sin u$

4) a. $\vec{r}(u,v)= u\hat{i}+v\hat{j}+\frac{v}{2}\hat{k}$ $x=u,\ y=v$
$z=\frac{v}{2}\Rightarrow z=\frac{y}{2}$

$z=y/2$ equation w/ no u or v plane

4) b. $\vec{r}(u,v) = 2u\cos(v)\hat{\imath} + 2u\sin(v)\hat{\jmath} + \frac{1}{2}u^2\hat{k}$

$\cos v, \sin v$ related by circle or ellipse equation

$$x^2+y^2=r^2$$
$$(2u\cos v)^2+(2u\sin v)^2=r^2$$
$$4u^2\cos^2 v+4u^2\sin^2 v=r^2$$
$$4u^2=r^2$$

$\frac{x^2+y^2}{4}=\frac{\cancel{4}u^2}{\cancel{4}}$ we also know that $z=\frac{1}{2}u^2 \Rightarrow 2z=u^2$

$\frac{x^2+y^2}{4}=2z$ or $x^2+y^2=8z$ or $z=\frac{1}{8}(x^2+y^2)$ paraboloid

c) $\vec{r}(u,v) = 3\cos v\cos u\,\hat{\imath} + 3\cos v\sin u\,\hat{\jmath} + 5\sin v\,\hat{k}$

this is an ellipse oriented a bit differently than we are used to w/ $z=5\sin v$

$x=3\cos v\cos u$

$y=3\cos v\sin u$

if we think of $v=\varphi$ $\sin\varphi$ & $\cos\varphi$ roles are switched this doesn't change the rectangular graph any, only where φ is measured from

$$\Rightarrow \frac{x^2}{9}+\frac{y^2}{9}+\frac{z^2}{25}=1$$

d) $\vec{r}(u,v)=4\cos u\,\hat{\imath}+4\sin u\,\hat{\jmath}+v\hat{k}$

$x^2+y^2=16$ this is a cylinder w/ z free

e) $\vec{r}(u,v)=u\hat{\imath}+v\hat{\jmath}+\sqrt{uv}\,\hat{k}$ let $x=u,\ y=v$

$$z=\sqrt{xy}$$

5. if $\vec{F}(x,y,z)$ is conservative, then $\vec{\nabla}f=\vec{F}$

given $\iint_S \vec{F}\cdot\vec{N}\,dS = \iiint_Q (\operatorname{div}F)\,dV$ according to the divergence theorem or $\iiint_Q \vec{\nabla}\cdot\vec{F}\,dV$ but since $\vec{F}=\vec{\nabla}f$ we have

$= \iiint_Q \vec{\nabla}\cdot\vec{\nabla}f\,dV = \iiint_Q \vec{\nabla}^2 f\,dV$ (see Del Notation handout)

6. mathematically speaking, you can think of an incompressible flow as a situation where the flux through a closed surface $= 0$ (there is as much going in as coming out)

a source has a positive flow out of a closed surface (think of this like a water hose in a bucket → more material is being added somewhere inside the surface

a sink has a negative flow out of a closed surface (or a positive one going in). think of this like a drain. less is coming out than is going in.

7. Based on part 5, it cannot be determined generally. To be incompressible, the ∇^2 would have to be zero and this need not be the case for conservative fields. Only if the field was already a curl of some other field could we be certain that the divergence (of the curl) was zero.

8. Answers will vary, but to give an example from astronomy, one could calculate the pressure of photons against a solar sail.