Math 2153, Practice Final Exam, Fall 2012 Name KEY

**Instructions:** Show all work. Answer all parts of each question. Use exact values unless specifically asked to round or do the problem numerically. Some problems will ask you to complete the question in its entirety and some to merely "set up". Read the instructions on the problem very carefully.

1. For the scalar function $f(x,y,z) = 3xy^3 + z^2 + x\tanh(z)$ and the vector field $\vec{G}(x,y,z) = \ln(xy)\vec{\imath} + z\vec{\jmath} - 4yx^3\vec{k}$, calculate the following:
   a. $\vec{\nabla}f$ (5 points)

$$\vec{\nabla}f = (3y^3 + \tanh z)\hat{\imath} + 9xy^2\hat{\jmath} + (2z + x\,\mathrm{sech}^2(z))\hat{k}$$

   b. $\vec{\nabla} \times \vec{G}$ (8 points)

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \ln x + \ln y & z & -4yx^3 \end{vmatrix} = (-4x^3 - 1)\hat{\imath} - (-12x^2y - 0)\hat{\jmath} + (0 - \tfrac{1}{y})\hat{k}$$

$$= (-4x^3 - 1)\hat{\imath} + 12x^2y\,\hat{\jmath} - \tfrac{1}{y}\hat{k}$$

   c. $\vec{\nabla} \cdot \vec{G}$ (5 points)

$$= \frac{1}{x} + 0 + 0 = \boxed{\frac{1}{x}}$$

   d. $\nabla[(\vec{\nabla}f) \cdot (\vec{\nabla} \times \vec{G})]$ (7 points)

$$\nabla\left[-12x^3y^3 - 4x^3\tanh z - 3y^3 - \tanh z + 108x^3y^3 - \frac{2z}{y} - \frac{x}{y}\mathrm{sech}^2 z\right]$$

$$\nabla\left[96x^3y^3 - 4x^3\tanh z - 3y^3 - \tanh z - \frac{2z}{y} - \frac{x}{y}\mathrm{sech}^2 z\right]$$

$$\boxed{= \left[288x^2y^3 - 12x^2\tanh z - \frac{1}{y}\mathrm{sech}^2 z\right]\hat{\imath} + \left[288x^3y^2 - 9y^2 + \frac{2z}{y^2} + \frac{x}{y^2}\mathrm{sech}^2 z\right]\hat{\jmath} + \left[-4x^3\mathrm{sech}^2 z - \mathrm{sech}^2 z + \frac{2x}{y}\mathrm{sech}^2 z\tanh z\right]\hat{k}}$$

2. Determine if the vector field $\vec{F}(x,y,z) = \left(-\frac{y}{x^2} - z\right)\vec{\imath} + \frac{1}{x}\vec{\jmath} + (2z - x)\vec{k}$ is conservative. If so, find the potential function. (10 points)

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\frac{y}{x^2} - z & \frac{1}{x} & 2z - x \end{vmatrix} = \underbrace{(0-0)}_{0}\hat{\imath} - \underbrace{(-1-(-1))}_{0}\hat{\jmath} + \underbrace{\left(-\frac{1}{x^2} - \left(-\frac{1}{x^2}\right)\right)}_{0}\hat{k}$$

$$= \vec{0} \quad \text{conservative}$$

$$\int -\frac{y}{x^2} - z \, dx = \frac{y}{x} - xz + C(y,z)$$

$$\int \frac{1}{x} \, dy = \frac{y}{x} + C(x,z)$$

$$\boxed{f(x,y,z) = \frac{y}{x} + z^2 - xz + K}$$

$$\int 2z - x \, dz = z^2 - xz + C(x,y)$$

3. Using the line integral $\int_C \rho(x,y,z)ds$ calculate the mass of the wire for $\rho(x,y,z) = |xy|$ and along the path $C: \vec{r}(t) = 4\cos(t)\vec{\imath} + 4\sin(t)\vec{\jmath} + 2t\vec{k}, k > 0, 0 \le t \le 4\pi$. (20 points)

$$r'(t) = -4\sin t\,\hat{\imath} + 4\cos t\,\hat{\jmath} + 2\hat{k}$$

$$\|r'(t)\| = \sqrt{16\sin^2 t + 16\cos^2 t + 4} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5} \qquad ds = 2\sqrt{5}\,dt$$

$$\rho(x,y,z) = xy \Rightarrow \rho(t) = 16\cos t \sin t$$

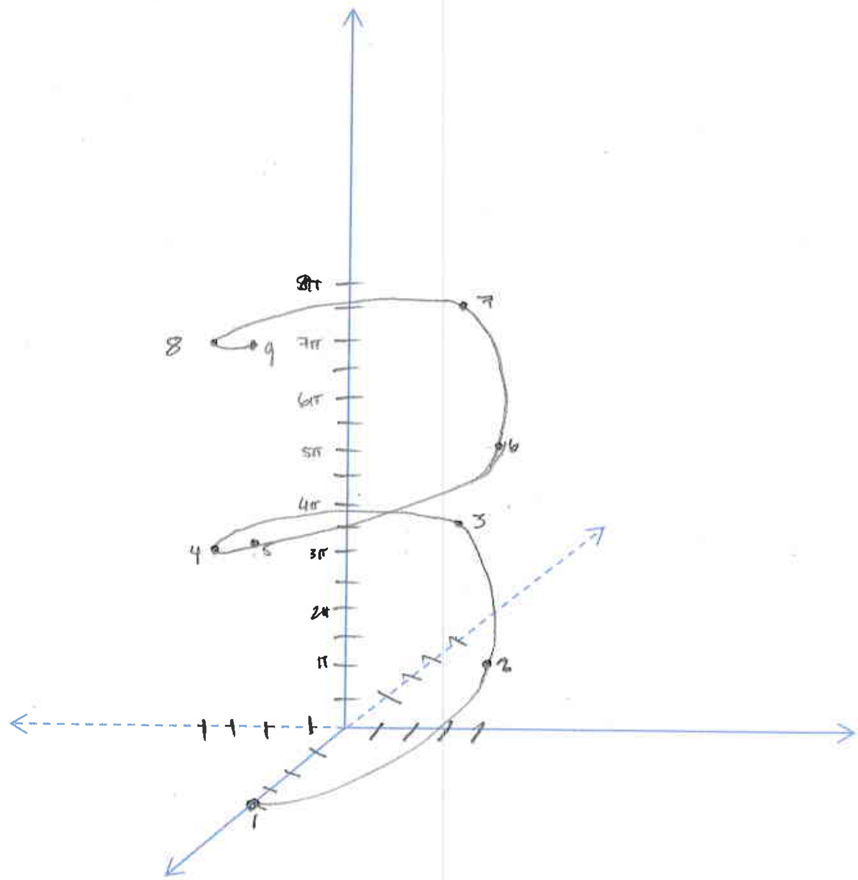
$$8\int_0^{\pi/2} 2\sqrt{5}\; 16\cos t \sin t \, dt = 8 \cdot 2\sqrt{5}\,\frac{1}{2}\sin^2 t \Big|_0^{\pi/2} = 8\sqrt{5}(1-0) = \boxed{8\sqrt{5}}$$

$u = \sin t$

$du = \cos t$

4. Sketch the path of the helix in Problem #3, by plotting the length of the wire measured in the line integral (2 turns of the helix). You should plot at least 9 points. You will want to scale your z-axis differently than your x- and y-axes. (10 points)

| $t$ | $x$ | $y$ | $z$ |
|---|---|---|---|
| $0$ | $4$ | $0$ | $0$ |
| $\pi/2$ | $0$ | $4$ | $\pi$ |
| $\pi$ | $-4$ | $0$ | $2\pi$ |
| $3\pi/2$ | $0$ | $-4$ | $3\pi$ |
| $2\pi$ | $4$ | $0$ | $4\pi$ |
| $5\pi/2$ | $0$ | $4$ | $5\pi$ |
| $3\pi$ | $-4$ | $0$ | $6\pi$ |
| $7\pi/2$ | $0$ | $-4$ | $7\pi$ |
| $4\pi$ | $4$ | $0$ | $8\pi$ |




5. Calculate the value of the line integral $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F}(x,y,z) = \left(\frac{y}{x^2} + y\right)\vec{\imath} + \left(\frac{-1}{x} + x\right)\vec{\jmath}$ along the path $C$: $circle\ (x-1)^2 + (y+3)^2 = 9$ clockwise from (4,-3) to (-2,-3). [Hint: is the field conservative?] (20 points)

$$\frac{\partial N}{\partial x} = \frac{1}{x^2} + 1 \qquad \frac{\partial M}{\partial y} = \frac{1}{x^2} + 1 \quad \checkmark$$

$$\int \frac{y}{x^2} + y \, dx = -\frac{y}{x} + xy + C(y)$$

$$\int \frac{-1}{x} + x \, dy = \frac{-y}{x} + xy + C(x)$$

$$f(x,y) = \frac{-y}{x} + xy + K$$

$$\int_C \vec{F} \cdot d\vec{r} = f(-2,-3) - f(4,-3) = \frac{3}{-2} + (-3)(-2) - \left[\frac{3}{4} + 3(-4)\right] =$$

$$-\frac{3}{2} - \frac{3}{4} + 6 + 12 = 18 - \frac{9}{4} = \boxed{\frac{63}{4}}$$

6. Calculate the value of the line integral $\int_C 2\arctan\left(\frac{y}{x}\right)dx + \ln(x^2+y^2)dy$ on the path $C$: *boundary of the region lying between* $x^2+y^2=1$ *and* $x^2+y^2=9$. (20 points)

$$\frac{\partial M}{\partial y} = 2\,\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{2x}{x^2+y^2}$$

$$\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x} = \frac{2y-2x}{x^2+y^2}$$

$$\frac{\partial N}{\partial x} = \frac{1}{x^2+y^2}\cdot 2y = \frac{2y}{x^2+y^2}$$

$$= \frac{2\cancel{r}\sin\theta - 2\cancel{r}\cos\theta}{r^{\cancel{2}}}$$

$$=$$

$$\int_0^{2\pi}\int_1^3 \cancel{\tfrac{1}{r}}\left[2\sin\theta - 2\cos\theta\right]\cancel{r}\,dr\,d\theta =$$

$$2\cdot 2\int_0^{2\pi} \sin\theta - \cos\theta\,d\theta = 4\left[-\cos\theta - \cancel{\sin\theta}^{\,0}\right]_0^{2\pi} = 4[-1-(-1)] = \boxed{0}$$

7. Consider the surface described parametrically by $\vec{r}(u,v) = u\cos(v)\vec{i} + 4v\vec{j} + u\sin(v)\vec{k}, 0 \le u \le \pi, 0 \le v \le 2\pi$. Find the following:

a. $\vec{N}$ to the surface, and evaluate it at the point $\left(\frac{1}{2},\pi,\frac{1}{2}\right)$. (10 points)

$$4v = \pi \Rightarrow v = \pi/4$$

$$\vec{r}_u = \cos v\,\hat{i} + 0\hat{j} + \sin v\,\hat{k}$$

$$u\cos\pi/4 = \tfrac{1}{2}$$

$$\vec{r}_v = -u\sin v\,\hat{i} + 4\hat{j} + u\cos v\,\hat{k}$$

$$u = \tfrac{1}{2}\cdot\sqrt{2} \qquad u = \tfrac{\sqrt{2}}{2}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & 0 & \sin v \\ -u\sin v & 4 & u\cos v \end{vmatrix} = (0-4\sin v)\hat{i} - \underbrace{(u\cos^2 v + u\sin^2 v)}_{u}\hat{j} + (4\cos v)\hat{k}$$

$$-4\sin v\,\hat{i} - u\hat{j} + 4\cos v\,\hat{k}$$

$$\vec{N}\left(\tfrac{1}{2},\pi,\tfrac{1}{2}\right) = -4\sin\pi/4\,\hat{i} - \tfrac{\sqrt{2}}{2}\hat{j} + 4\cos\pi/4\,\hat{k}$$

$$= \boxed{-2\sqrt{2}\,\hat{i} - \tfrac{\sqrt{2}}{2}\hat{j} + 2\sqrt{2}\,\hat{k}}$$

b. The equation of the tangent line at the given point, and the equation of the normal line at the same point. (5 points)

$$-2\sqrt{2}\left(x-\tfrac{1}{2}\right) - \tfrac{\sqrt{2}}{2}(y-\pi) + 2\sqrt{2}\left(z-\tfrac{1}{2}\right) = 0$$ tangent plane

$$\vec{r}(t) = \left(-2\sqrt{2}t + \tfrac{1}{2}\right)\hat{\imath} + \left(-\tfrac{\sqrt{2}}{2}t + \pi\right)\hat{\jmath} + \left(2\sqrt{2}t + \tfrac{1}{2}\right)\hat{k}$$

or $$\frac{x-1/2}{-2\sqrt{2}} = \frac{y-\pi}{(\sqrt{2}/2)} = \frac{z-1/2}{2\sqrt{2}}$$ normal line

c. Find the area of the surface with the given limits in u and v. (10 points)

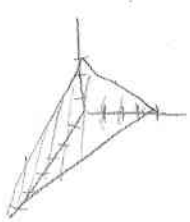
$$\int_0^{2\pi}\int_0^{\pi}\sqrt{16\sin^2 v + u^2 + 4\cos^2 v}\,du\,dv =$$

$$\int_0^{\pi}\int_0^{2\pi}\sqrt{16+u^2}\,dv\,du = 2\pi\int_0^{\pi}\sqrt{16+u^2}\,du \rightarrow$$

Can complete The rest numerically

$$\approx 2\pi(13.7592) \approx \boxed{86.4517}$$

8. Evaluate the flux integral $\iint_S \vec{F}\cdot\vec{N}dS$, for the field $\vec{F}(x,y,z) = 3y\vec{\imath} - 4z\vec{\jmath} - x\vec{k}$ and for the surface $S$: $2x+3y+4z=12$, in the first octant. (20 points)



$G(x,y,z) = 2x+3y+4z-12=0$

$\nabla G = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$

$z=0$ $\quad z = \frac{12-2x-3y}{4}$

$2x+3y=12$

$y = \frac{12-2x}{3} = 4 - \frac{2}{3}x$

$y=0 \Rightarrow x=6$

$$\iint_R 3y\cdot 2 - \cancel{4}\left(\frac{12-2x-3y}{\cancel{4}}\right)\cdot 3 - 4x\,dA = \int_0^6\int_0^{4-\frac{2}{3}x} 15y + 2x - 36\,dy\,dx$$

$6y - 36 + 6x + 9y - 4x$

$$\int_0^6 \frac{15}{2}y^2 + 2xy - 36y\Big|_0^{4-\frac{2}{3}x} dx =$$

$120 \quad -40x + \frac{10}{3}x^2 \quad 8x - \frac{4}{3}x \quad -144 + 24x$

$$\int_0^6 \frac{15}{2}\left(16 - \frac{16}{3}x + \frac{4}{9}x^2\right) + 2x\left(4-\frac{2}{3}x\right) - 36\left(4-\frac{2}{3}x\right)dx = \int_0^6 -24 - 8x + 2x^2\,dx =$$

$$-24x - 4x^2 + \frac{2}{3}x^3\Big|_0^6 = -144$$

9. Recalculate the flux integral in problem #8 using the Divergence Theorem. (12 points)

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}[3y] + \frac{\partial}{\partial y}[-4z] + \frac{\partial}{\partial z}[-x] = 0$$

$$\int_0^6 \int_0^{4-\frac{2}{3}x} \int_0^{\frac{12-2x-3y}{4}} 0 \; dz\,dy\,dx = \boxed{0}$$

The answers to #8 & #9 differ because the divergence Theorem calculates over a closed surface & #8 only calculated the plane $2x+3y+4z=12$ if we also calculated the $\vec{N}$ vectors for the 3 remaining sides ($-\hat{i}, -\hat{j}, -\hat{k}$) then the sum of the integral would all cancel out.

10. Use Stokes' Theorem to calculate the line integral $\int_C \vec{F}\cdot d\vec{r}$ for the field $\vec{F}(x,y,z) = 2y\vec{i} + +3z\vec{j} + xk$ along the path $C$: *triangle with vertices* (0,0,0), (0,1,2), (1,1,1). (20 points)

S: $\hat{j}+2\hat{k}$ , $\hat{i}+\hat{j}+\hat{k}$

hint: 2 vectors in plane, X gives normal
see Chapter 11.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = (1-2)\hat{i} + 2\hat{j} + (0-1)\hat{k}$$

$-1\hat{i} + 2\hat{j} - \hat{k}$

$-1(x-0)+2(y-0)-1(z-0)=0$

$-x+2y-z=0$

$\cancel{z}-2y+x=0=G(x,y,z)$

$\nabla G = 1\hat{i} - 2\hat{j} + \hat{k}$

$$\vec{\nabla}\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y & 3z & x \end{vmatrix} = (0-3)\hat{i}-(1-0)\hat{j}+(0-2)\hat{k}$$

$-3\hat{i}-\hat{j}-2\hat{k}$

$z=0$

$-2y+x=0$

$2y=x$

$y=\frac{1}{2}x$

$$(\vec{\nabla}\times\vec{F})\cdot\vec{\nabla}G = \langle -3,-1,-2\rangle\cdot\langle 1,-2,1\rangle = -3+\cancel{2}-\cancel{2} = -3$$

$$\int_0^1\int_0^{\frac{1}{2}x} -3\,dy\,dx = \int_0^1 -3y\Big|_0^{\frac{1}{2}x} = \int_0^1 -\frac{3}{2}x\Big|_0^1 = -\frac{3}{4}x^2\Big|_0^1 = \boxed{-\frac{3}{4}}$$

11. Find an equation for the curvature of the cycloid $\vec{r}(t) = cos^3 t\vec{\imath} + 2sin^3 t\vec{\jmath}$. (8 points)

$r'(t) = -3\cos^2 t \sin t\, \hat{\imath} + 6\sin^2 t \cos t\, \hat{\jmath}$

$r''(t) = (6\cos t \sin^2 t + 3\cos^3 t)\hat{\imath} + (12\sin t \cos^2 t - 6\sin^3 t)\hat{\jmath}$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3\cos^2 t \sin t & 6\sin^2 t \cos t & 0 \\ 6\cos t \sin^2 t + 3\cos^3 t & 12\sin t\cos^2 t - 6\sin^3 t & 0 \end{vmatrix} = 0\hat{\imath} + 0\hat{\jmath} + ($$

$(-36\cos^4 t \sin^2 t + 18\cos^2 t \sin^4 t - 36\sin^4 t\cos^2 t - 18\sin^2 t\cos^4 t)\hat{k}$

$= 18\cos^2 t\sin^2 t(-2\cos^2 t - 2\sin^2 t + \sin^2 t\cos^2 t) = 18\cos^2 t\sin^2 t(-2+1) =$

$-18\cos^2 t\sin^2 t = \|r'(t) \times r''(t)\|$

$\|r'(t)\| = \sqrt{9\cos^4 t\sin^2 t + 36\sin^4 t\cos^2 t} = \sqrt{9\cos^2 t\sin^2 t(\cos^2 t + 4\sin^2 t)} = 3\cos t\sin t\sqrt{1+3\sin^2 t}$

$1+3\sin^2 t$

$K = \frac{\cancel{18}^{6}\cancel{\cos^2 t\sin^2 t}}{\cancel{3}\cancel{\cos t\sin t}\sqrt{1+3\sin^2 t}}$

$= \boxed{\frac{6}{\sqrt{1+3\sin^2 t}}}$

12. Determine if $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+4y^2}$ exists or does not exist. (10 points)

let $x=0$ $\lim_{y\to 0} \frac{0}{4y^2} = 0$

let $y=0$ $\lim_{x\to 0} \frac{0}{x^6} = 0$

let $x=y$ $\lim_{x\to 0} \frac{x^3 \cdot x}{x^6+4x^2} = \frac{\cancel{x^4}^{2}}{\cancel{x^2}(x^4+4)} = \frac{x^2}{(x^4+4)} = 0$

let $y=kx^3$ $\lim_{x\to 0} \frac{x^3 \cdot kx^3}{x^6+4k^2x^6} = \frac{k\cancel{x^6}}{\cancel{x^6}(1+4k^2)} = \frac{k}{1+4k^2} \neq 0$ if $k \neq 0$

DNE

13. Find the partial derivative $f_{zxyz}$ for the function $f(x,y,z) = x^3\ln(y^z)$. (12 points)

$= x^3 z\ln y$

$f_z = x^3 \ln y$

$f_{zx} = 3x^2 \ln y$

$f_{zxy} = \frac{3x^2}{y}$

$f_{zxyz} = 0$

14. Find the normal vector to the graph $z = x^3y^2 + 1$ at the point (1,2,5). Is this vector oriented to the outward surface or the inward surface? Then give the equation of the tangent plane. (15 points)

$F(x,y,z) = x^3y^2+1-z$

$\nabla F = 3x^2y^2\hat{i} + 2x^3y\,\hat{j} - 1\hat{k}$

$\nabla F(1,2,5) = 12\hat{i} + 4\hat{j} - 1\hat{k}$

inward

outward $G(x,y,z) = z - x^3y^2 - 1$

$\nabla G = -3x^2y^2\hat{i} - 2x^3y\,\hat{j} + 1\hat{k}$

$\nabla G(1,2,5) = -12\hat{i} - 4\hat{j} + \hat{k}$

$12(x-1) + 4(y-2) - 1(z-5) = 0$

or

$-12(x-1) - 4(y-2) + 1(z-5) = 0$

15. Find any critical points for the graph $f(x,y) = 12 - 4x - 2y - x^2 + y^2$. Characterize the critical points as a relative maximum, relative minimum, a saddle point, or cannot be determined. (12 points)

$f_x = -4 - 2x = 0 \quad -4 = 2x \quad x = -2$

$f_y = -2 + 2y = 0 \quad -2 = -2y \quad y = 1$

$\boxed{(-2,1)}$

$f_{xx} = -2$

$f_{yy} = 2$

$f_{xy} = 0$

$f_{xx}f_{yy} - (f_{xy})^2 =$

$(-2)(2) - 0 = -4 < 0$ $\boxed{\text{saddle point}}$

16. Integrate. (7 points each)

a. $\int_0^4 \int_1^2 x^3 - 2y^{-1} + 7\,dxdy$

1

$$\frac{1}{4}x^4 - 2y^{-1}x + 7x \Big|_1^2 = 4 - \frac{4}{y} + 14 - \frac{1}{4} + \frac{2}{y} - 7 = \frac{43}{4} - \frac{2}{y}$$

$$\int_1^4 \frac{43}{4} - \frac{2}{y}\,dy = \frac{43}{4}y - 2\ln y \Big|_1^4 = \frac{43}{\cancel{4}}\cancel{4} - 2\ln 4 - \frac{43}{4} + \cancel{2\ln 1}^{0}$$

$$\boxed{\frac{129}{4} - 2\ln 4}$$

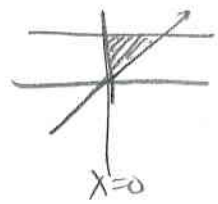
intersection

y=2

b. $\int_0^2 \int_x^2 x\sqrt{1+y^3}\,dydx$ [Hint: you will want to change the order of integration.]

x=0 y=x

$$\int_0^2 \int_0^y x\sqrt{1+y^3}\,dx\,dy =$$




$$\int_0^2 \frac{1}{2}x^2\sqrt{1+y^3}\Big|_0^y dy = \int_0^2 \frac{1}{2}y^2\sqrt{1+y^3}\,dy$$

$u = 1+y^3$

$du = 3y^2\,dy$

$\frac{1}{3}du = y^2\,dy$

$$\frac{1}{6}\int_1^9 u^{1/2}\,du = \frac{1}{\cancel{6}_3}u^{3/2}\cdot\frac{\cancel{2}}{3}\Big|_1^9 =$$

$$\frac{1}{9}(9^{3/2} - 1^{3/2}) = \frac{1}{9}(27-1) = \boxed{\frac{26}{9}}$$

c. $\int_0^4 \int_0^{\pi/2} \int_0^{1-x} z\sin(x)\,dzdxdy$

$$\frac{z^2}{2}\Big|_0^{1-x} \qquad \frac{1}{2}(1-x)^2 = \frac{1}{2}(1-2x+x^2)$$

$$\frac{1}{2}\int_0^4 \int_0^{\pi/2} (1-2x+x^2)\sin x\,dx\,dy$$

| | u | dv |
|---|---|---|
| + | $1-2x+x^2$ | $\sin x$ |
| − | $-2+2x$ | $-\cos x$ |
| + | 2 | $-\sin x$ |
| − | 0 | $\cos x$ |

$$\frac{1}{2}\int_0^4 \left[(1-2x+x^2)(-\cos x) - (-2+2x)(-\sin x) + 2\cos x\right]_0^{\pi/2} dy =$$

$$\frac{1}{2}\int_0^4 \cancel{(1-\pi+\pi^2/4)(0)}^{0} - (1)(-1) + (-2+\pi)(+1) + \cancel{(-2)(0)}^{0} + 2(0) - 2(1)\,dy =$$

$$\frac{1}{2}\int_0^4 1 - 2 + \pi - 2\,dy = \frac{1}{2}\int_0^4 -3+\pi\,dy = \frac{1}{2}(4)(-3+\pi) = \boxed{-6+2\pi}$$

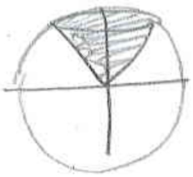
17. Find the volume of the region bounded by the cone $z = \sqrt{\frac{x^2+y^2}{5}}$, and the sphere $x^2 + y^2 + z^2 = 25$. (15 points)

$\rho = 5$

$\sqrt{5}\,\cancel{\rho}\cos\varphi = \cancel{\rho}\sin\varphi$

$\sqrt{5} = \tan\varphi$

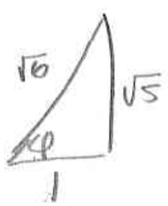
$\varphi = \tan^{-1}(\sqrt{5})$



$$\int_0^{2\pi}\int_0^{\tan^{-1}(\sqrt{5})}\int_0^{5} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\frac{1}{3}\rho^3 \Big|_0^5 = \frac{125}{3}$$

$$\int_0^{2\pi}\int_0^{\tan^{-1}(\sqrt{5})} \frac{125}{3}\sin\varphi \, d\varphi \, d\theta = \int_0^{2\pi} -\frac{125}{3}\cos\varphi \Big|_0^{\tan^{-1}\sqrt{5}} d\theta$$




$$-\frac{125}{3}\int_0^{2\pi} \cos(\tan^{-1}\sqrt{5}) - 1 \, d\theta = -\frac{125}{3}\int_0^{2\pi}\left(\frac{1}{\sqrt{6}} - 1\right) d\theta =$$

$$\left(1 - \frac{1}{\sqrt{6}}\right)\frac{125}{3}\,\theta \Big|_0^{2\pi} = \boxed{\left(1 - \frac{1}{\sqrt{6}}\right)\frac{125}{3}\cdot 2\pi}$$