Math 2568, Exam #2, Part 1, Fall 2013 Name ______________________

Instructions: On this portion of the exam, you may **NOT** use a calculator. Show all work. Answers must be supported by work to receive full credit.

1. Compute the determinant by the cofactor method. (10 points)

$$\begin{vmatrix} 1 & 3 & -1 & 2 & 3 \\ 0 & 1 & 1 & 3 & -4 \\ 0 & 0 & 0 & -2 & 4 \\ 2 & 5 & 2 & 1 & 0 \\ 0 & 0 & -4 & 0 & 1 \end{vmatrix}$$

$$1\begin{vmatrix} 1 & 1 & 3 & -4 \\ 0 & 0 & -2 & 4 \\ 5 & 2 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & -1 & 2 & 3 \\ 1 & 1 & 3 & -4 \\ 0 & 0 & -2 & 4 \\ 0 & -4 & 0 & 1 \end{vmatrix} = (1)(1)\begin{vmatrix} 0 & -2 & 4 \\ 2 & 1 & 0 \\ -4 & 0 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 & -4 \\ 0 & -2 & 4 \\ -4 & 0 & 1 \end{vmatrix}$$

$$-2\cdot 3\begin{vmatrix} 1 & 3 & -4 \\ 0 & -2 & 4 \\ -4 & 0 & 1 \end{vmatrix} - 2(-1)\begin{vmatrix} -1 & 2 & 3 \\ 0 & -2 & 4 \\ -4 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 2 & 0 \\ -4 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ -4 & 0 \end{vmatrix} + 5\begin{vmatrix} -2 & 4 \\ 0 & 1 \end{vmatrix}$$

$$+5(-4)\begin{vmatrix} 3 & -4 \\ -2 & 4 \end{vmatrix} - 6\begin{vmatrix} -2 & 4 \\ 0 & 1 \end{vmatrix} - 6(-4)\begin{vmatrix} 3 & -4 \\ -2 & 4 \end{vmatrix} + 2(-1)\begin{vmatrix} -2 & 4 \\ 0 & 1 \end{vmatrix}$$

$$+2(-4)\begin{vmatrix} 2 & 3 \\ -2 & 4 \end{vmatrix} = 2(2) + 4(4) + 5(-2) - 20(12-8) - 6(-2)$$

$$+24(12-8) - 2(-2) - 8(8+6) = 4 + 16 - 10 - 80 + 12 + 96$$

$$+4 - 112$$

$$= -70$$

2. Compute the determinant by using row operations. (7 points)

$$\begin{vmatrix} 0 & -3 & 2 & 0 \\ 2 & -1 & 2 & 0 \\ 4 & 3 & 1 & 1 \\ -1 & 0 & 0 & -5 \end{vmatrix}$$

$2R_4 + R_2 \to R_2$

$4R_4 + R_3 \to R_3$

$$\begin{vmatrix} 0 & -3 & 2 & 0 \\ 0 & -1 & 2 & -10 \\ 0 & 3 & 1 & -19 \\ -1 & 0 & 0 & -5 \end{vmatrix} = -(-1)\begin{vmatrix} -3 & 2 & 0 \\ -1 & 2 & -10 \\ 3 & 1 & -19 \end{vmatrix}$$

$R_1 + R_3 \to R_3$

$$\begin{vmatrix} -3 & 2 & 0 \\ -1 & 2 & -10 \\ 0 & 3 & -19 \end{vmatrix}$$

$-\frac{1}{3}R_1 + R_2 \to R_2$

$$\begin{vmatrix} -3 & 2 & 0 \\ 0 & 4/3 & -10 \\ 0 & 3 & -19 \end{vmatrix}$$

$$= -3\begin{vmatrix} 4/3 & -10 \\ 3 & -19 \end{vmatrix}$$

$3R_1 \to R_1$

$$\frac{-3}{3}\begin{vmatrix} 4 & -30 \\ 3 & -19 \end{vmatrix} =$$

$$-1(-76 + 90) = -1(14) =$$

$$= -14$$

3. Given that A and B are $n \times n$ matrices with det A = 5 and det B = -2, find the following. (3 points each)

a) det (BA) -10

b) det (B^{-1}) $-\frac{1}{2}$

c) det (AB^k) $5(-2)^k$

d) det ($-2B^2$) $(-2)^n(-2)^2 = (-2)^{n+2}$

e) det (kA) $k^n \cdot 5$

f) det ($A^{-1}BA$) -2

4. Suppose that det(C)=10. Find the determinant of the matrix after the following row operations. (6 points)

$$\underbrace{-R_1 + R_2 \to R_2}_{\text{no change}}, \underbrace{R_1 \leftrightarrow R_3}_{(-1)}, \underbrace{2R_2 - 3R_4 \to R_4}_{\text{change by } (-3)}, \underbrace{\frac{1}{2}R_4 \to R_4}_{\text{change by } 1/2}$$

$10(-1)(-3)(1/2) = \frac{30}{2} = \boxed{15}$

5. Determine if the following spaces is a subspace of P_n. If it is, prove it. If it is not, find an example to the contrary. (6 points each)

a. All polynomials of the form $p(t) = a + bt + ct^2$

yes. it is.

i) $a = b = c = 0$ $p(t) = 0$ ✓

ii) adding $p(t) + g(t) = (a+d) + (b+e)t + (c+f)t^2$ ✓ $a+d$, $b+e$, $c+f$ real

iii) scalar $kp(t) = (ka) + (kb)t + (kc)t^2$ ✓ ka, kb, kc real

b. All polynomials of the form $p(t) = a_0 + a_1t + a_6t^6 + a_{11}t^{11}$.

yes, it is.

i) $a_0 = a_1 = a_6 = a_{11} = 0$ ✓

ii) adding $p(t) + g(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_6 + b_6)t^6 + (a_{11} + b_{11})t^{11}$

$a_i + b_i$ real.

iii) scalar $kp(t) = ka_0 + ka_1t + ka_6t^6 + ka_{11}t^{11}$

ka_i real ✓

6. Determine if each statement is True or False. (2 points each)

a. T F Det(-1A) is always equal to -1Det(A).

b. T F If the determinant of a matrix is equal to zero, then the columns of the matrix are linearly dependent.

c. T F The determinant of a triangular matrix is the difference of the entries on the diagonal.

d. T F If $\{\vec{u}, \vec{v}, \vec{w}, \vec{x}, \vec{y}, \vec{z}\}$ is linearly independent, then so is $\{\vec{u}, \vec{v}, \vec{x}, \vec{z}\}$.

e. T F If A and B are $m \times n$ matrices, then both $\det(AB^T)$ and $\det(A^T B)$ are defined.

m×n · n×m = m×m   n×m · n×m = n×n

f. T F $\{\vec{0}\}$ is a subspace.

g. T F If two spaces have the same number of basis vectors, then then are isomorphic.

h. T F The pivot rows of a matrix are always linearly independent.

i. T F The null space of an mxn matrix is a subspace of R^m.

j. T F If A and B are row equivalent, then their column spaces are the same.

k. T F The vector space $\mathbb{P}_n$ and R^{n+1} are isomorphic.

l. T F A linearly dependent set in a subspace H that spans the space is a basis for H.

m. T F The kernel of a matrix is a subspace of the domain of the matrix.

n. T F The coordinates of a position in space are the same regardless of the basis.

o. T F An isomorphism is a mapping that is both one-to-one and onto.

p. T F The third standard basis vector $\vec{e_3}$ in P_6 is t^2.

7. Describe in your own words how Nul A and Col A are related to each other (or contrast with each other). (6 points)

the number of vectors in Nul A + number of vectors in Col A = n. Nul A is in domain, but Col A is in the codomain. or any other comparison from the chart in our textbook.

Math 2568, Exam #2, Part 2, Fall 2013 Name KEY

Instructions: On this portion of the exam, you *may* use a calculator to perform elementary matrix operations. Support your answers with work (reproduce the reduced matrices from your calculator) or other justification for full credit.

1. Determine if the following sets are linearly independent or dependent. If the sets are dependent, find a basis for the subspace spanned by the vectors. Is the set a basis for the entire vector space? ($\mathbb{R}^5$ or $\mathbb{R}^3$ or $\mathbf{P}_3$ respectively) (4 points each)

a. $\left\{\begin{bmatrix}0\\-1\\1\\1\\9\end{bmatrix},\begin{bmatrix}2\\2\\-2\\0\\7\end{bmatrix},\begin{bmatrix}5\\1\\0\\2\\-4\end{bmatrix},\begin{bmatrix}4\\1\\-3\\-2\\0\end{bmatrix},\begin{bmatrix}3\\6\\-2\\-8\\19\end{bmatrix}\right\}$ reduces to I

it is a basis

b. $\left\{\begin{bmatrix}1\\-1\\-2\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}-2\\2\\-5\end{bmatrix}\right\}$ not a basis.

zero vector

does not span
not independent

c. $\{4-t, 6t+t^2-t^3, t^2+1\}$

only 3 vectors

P_3 needs 4

not a basis

does not span

2. Given the basis for $\mathbb{R}^3$ to be $\mathbf{B}=\left\{\begin{bmatrix}2\\0\\5\end{bmatrix},\begin{bmatrix}-1\\1\\4\end{bmatrix},\begin{bmatrix}3\\-2\\0\end{bmatrix}\right\}$, state P_B matrix and find the representation of $\vec{x}=\begin{bmatrix}3\\-1\\9\end{bmatrix}$ in the new basis. (5 points)

$$P_B=\begin{bmatrix}2&-1&3\\0&1&-2\\5&4&0\end{bmatrix}$$

$$\vec{x}=P_B[\vec{x}]_B$$

$$[x]_B=P_B^{-1}\vec{x}=\begin{bmatrix}3/11\\21/11\\16/11\end{bmatrix}$$

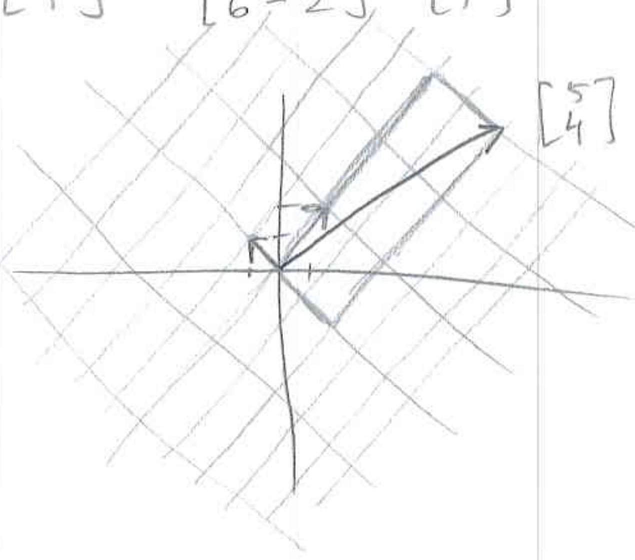
3. Given the vector $[\vec{x}]_B=\begin{bmatrix}7\\-2\\3\\1\end{bmatrix}$, find the vector $\vec{x}$ in the standard basis given the basis $\mathbf{B}=\left\{\begin{bmatrix}2\\2\\0\\0\end{bmatrix},\begin{bmatrix}1\\3\\4\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\\5\end{bmatrix},\begin{bmatrix}-1\\2\\0\\1\end{bmatrix}\right\}$. (5 points)

$$P_B=\begin{bmatrix}2&1&1&-1\\2&3&0&2\\0&4&1&0\\0&0&5&1\end{bmatrix}$$

$$P_B[\vec{x}]_B=\vec{x}=\begin{bmatrix}14\\10\\-5\\16\end{bmatrix}$$

4. Graphically interpret the meaning of the following information: A basis for R^2 is $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$; the coordinates for $[\vec{x}]_B=\begin{bmatrix}3\\-2\end{bmatrix}$. Sketch the situation and find the vector $\vec{x}$ in the standard basis. (7 points)

$$3\begin{bmatrix}1\\2\end{bmatrix}-2\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}3+2\\6-2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}$$



5. Given the matrix $A = \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 0 & 3 & 4 & 1 & 1 & -1 \\ 2 & -3 & 2 & 1 & 1 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 \end{bmatrix}$, find the following:

a. Find an explicit description of Nul A. (7 points)

rref $\Rightarrow$ $\begin{bmatrix} 1 & 0 & 0 & 0 & -23/14 & -33/56 \\ 0 & 1 & 0 & 0 & -5/7 & -9/28 \\ 0 & 0 & 1 & 0 & 1/2 & -1/8 \\ 0 & 0 & 0 & 1 & 8/7 & 13/28 \end{bmatrix}$ $\underbrace{\qquad\qquad}_{\text{free}}$

$x_1 = 23/14\, x_5 + 33/56\, x_6$

$x_2 = 5/7\, x_5 + 9/28\, x_6$

$x_3 = -1/2\, x_5 + 1/8\, x_6$

$x_4 = -8/7\, x_5 - 13/28\, x_6$

$x_5 = x_5$

$x_6 = x_6$

$\vec{x} = t \begin{bmatrix} 23 \\ 10 \\ -7 \\ -16 \\ 14 \\ 0 \end{bmatrix} + s \begin{bmatrix} 33 \\ 18 \\ 7 \\ -26 \\ 0 \\ 56 \end{bmatrix}$

$\text{Nul } A = \left\{ \begin{bmatrix} 23 \\ 10 \\ -7 \\ -16 \\ 14 \\ 0 \end{bmatrix}, \begin{bmatrix} 33 \\ 18 \\ 7 \\ -26 \\ 0 \\ 56 \end{bmatrix} \right\}$

b. Find a basis for Col A. (4 points)

$\text{Col } A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$

6. For each of the following questions, provide a short explanation with theoretical justifications. (4 points each)

a. What is a triangular matrix? Define it and give an example.

a matrix whose entries above or below the main diagonal are all zero

e.g. $\begin{bmatrix} 4 & 5 & 7 & 1 \\ 0 & 6 & 8 & 2 \\ 0 & 0 & 9 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ or $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 2 & 6 & 8 & 0 \\ 1 & 7 & 9 & 1 \end{bmatrix}$

b. Explain why R^2 is **NOT** a subspace of R^3.

every element of $\mathbb{R}^3$ has 3 coordinates and is a linear combination of the $\mathbb{R}^3$ bases vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$. $\mathbb{R}^2$ has only 2 coordinates and is a linear combination of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$. $\mathbb{R}^2$ is isomorphic to a plane in $\mathbb{R}^3$ but isomorphic is not "identical to".

c. What does the Spanning Set Theorem say? Give an example of its use.

This Theorem says that if you have a set of vectors that spans a space (or subspace), you can remove any dependent vectors from the set w/o changing the span. From this process you can find a basis for the space by removing all dependent vectors.