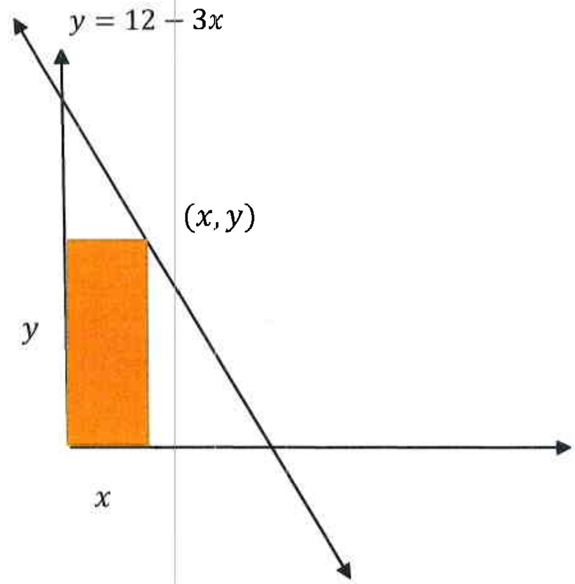
Math 1151, Exam #3, Fall 2014 Name KEY

Instructions: Show all work. Answers without work may only receive partial credit. If you are asked for an explanation, explain as completely as possible. Use exact answers unless specifically asked to round.

1. A rectangle is constructed with one side on the positive x-axis and one side on the positive y-axis, and the vertex on the line $y = 12 - 3x$. What dimensions maximize the area of the rectangle? What is the maximum area? (8 points)



$A = x(12-3x) = 12x - 3x^2$

$A' = 12 - 6x = 0$

$12 = 6x$

$x = 2$

$y = 12 - 3x = 12 - 3(2) = 12 - 6 = 6$

2×6

max area is 12

2. a. Write an equation that represents the linear approximation to the function $f(x) = \sqrt[4]{x}$ at the point $a = 81$. (6 points)

$f(x) = x^{1/4}$ $\quad f'(x) = \frac{1}{4}x^{-3/4} = \frac{1}{4} \cdot \frac{1}{\sqrt[4]{x^3}}$ $\quad f(81) = 3$

$f'(81) = \frac{1}{4} \cdot \frac{1}{(\sqrt[4]{81})^3} = \frac{1}{4} \cdot \frac{1}{3^3} = \frac{1}{4} \cdot \frac{1}{27} = \frac{1}{108}$

$L(x) = 3 + \frac{1}{108}(x - 81)$

b. Use the resulting equation to approximate the value of $f(85)$. (2 points)

$L(85) = 3 + \frac{1}{108}(85 - 81) = 3 + \frac{1}{108}(4) = 3 + \frac{1}{27} = \frac{82}{27}$

≈ 3.037

c. Calculate the % error using the exact value obtained from your calculator. (2 points)

$$\frac{\frac{82}{27} - \sqrt[4]{85}}{\sqrt[4]{85}} = 2.1959 \times 10^{-4}$$

$$\approx .022\%$$

3. Determine whether Rolle's Theorem applies to the function $f(x) = x^3 - 2x^2 - 8x$ on the interval $[-2,4]$, and if it does, find the point(s) guaranteed to exist. (7 points)

$f(-2) = -8 - 2(4) + 16 = 0$

$f(4) = 64 - 2(16) - 32 = 0$

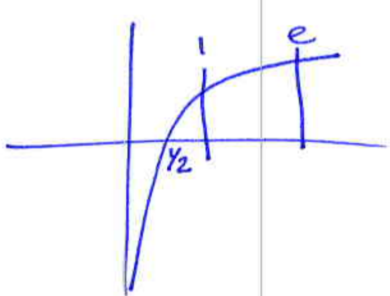
yes, it applies since $f(a) = f(b)$. and $f(x)$ is continuous and differentiable

$f'(x) = 3x^2 - 4x - 8 = 0$

$$c = \frac{4 \pm \sqrt{16+96}}{6} = \frac{4 \pm \sqrt{112}}{6} = \frac{4 \pm 2\sqrt{28}}{6} = \frac{2 \pm 2\sqrt{7}}{3}$$

at least one of these must be on $[-2,4]$

4. Make a sketch of the function $f(x) = \ln(2x)$ on the interval $[1, e]$. Determine whether the Mean Value Theorem applies on the given interval. If so, find the point(s) guaranteed to exist by the theorem. (8 points)



yes it applies

$f(x)$ is continuous on $[1,e]$

and differentiable on $(1,e)$

$$m = \frac{\ln(2e) - \ln(2)}{e-1} = \frac{\cancel{\ln 2} + \ln e - \cancel{\ln 2}}{e-1} = \frac{1}{e-1}$$

$$\frac{1}{e-1} = \frac{1}{2x} \cdot 2 \Rightarrow \frac{1}{e-1} = \frac{1}{x} \Rightarrow x = e-1$$

5. Use L'Hôpital's Rule to find the indicated limits. Be sure to check that L'Hôpital's applies before proceeding. (6 points each)

a. $\lim_{x\to 0} \frac{e^x-1}{x^2+3x}$

$\frac{1-1}{0} = \frac{0}{0}$

$\lim_{x\to 0} \frac{e^x}{2x+3} = \frac{1}{3}$

b. $\lim_{x\to\infty} \frac{\tan^{-1}x-\frac{\pi}{2}}{1/x}$

$\frac{\frac{\pi}{2}-\frac{\pi}{2}}{1/\infty} = \frac{0}{0}$

$\lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{\frac{-1}{x^2}} \frac{x^2}{x^2} = \frac{-x^2}{1+x^2} = \frac{\infty}{\infty}$

$\lim_{x\to\infty} \frac{-2x}{2x} = -1$

c. $\lim_{x\to 0} x\csc x = \lim_{x\to 0} \frac{x}{\sin x} = \frac{0}{0}$

$\lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{1} = 1$

d. $\lim_{x\to 0^+} x^{2x} = 0^0 = L$

$\lim_{x\to 0^+} \ln x^{2x} = \ln L = \lim_{x\to 0^+} 2x\ln x =$

$\lim_{x\to 0^+} \frac{2\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{2/x}{-1/x^2} \frac{x^2}{x^2} = \lim_{x\to 0^+} \frac{2x}{-1} = 0$

$\ln L = 0 \Rightarrow L = 1$

e. $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{1/x} = 1^0 = 1$ not indeterminant

$\lim_{x\to\infty} \frac{1}{x}\ln\left(1+\frac{a}{x}\right) = \ln L \overset{?}{=} \lim_{x\to\infty} \frac{\ln\left(1+\frac{a}{x}\right)}{x} = \frac{0}{\infty} = 0$

$e^0 = 1$

6. Use Newton's Method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ to approximate the zero of the function starting from $x_0 = -2$. (7points)

use $f(x) = \ln(x^2 - e^x)$

$f'(x) = \frac{1}{x^2 - e^x}(2x - e^x) = \frac{2x - e^x}{x^2 - e^x}$

$X_0 = -2$

$X_1 = -.7366$

$X_2 = -.8266$

$X_3 = -.991576$

$X_4 = -1.11917...$

$X_5 = -1.1469...$

$X_6 = -1.1477...$

$X_7 = -1.147757632 = X_8$

Zero is $X = -1.147757632$

7. Find the antiderivatives for the following functions. (6 points each)

a. $\int 3x^{-2} - 4x^2 + 8e^{7x} + 1\, dx$

$-3x^{-1} - \frac{4}{3}x^3 + \frac{8}{7}e^{7x} + x + C$

b. $\int \frac{12t^8 - t}{t^3} dt = \int 12t^5 - t^{-2} dt$

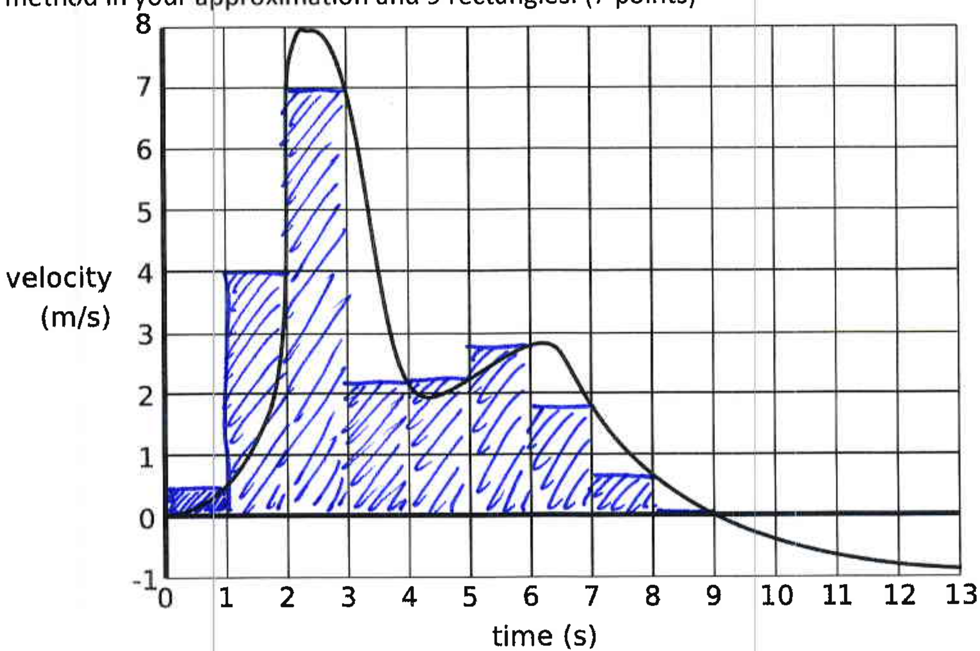
$= 2t^6 + \frac{1}{t} + C$

c. $\int \sec 4\theta \tan 4\theta\, d\theta$

$\frac{1}{4}\sec 4\theta + C$

d. $\int \frac{6}{\sqrt{25-x^2}}\,dx$ $\quad 6\sin^{-1}\left(\frac{x}{5}\right)+C$

8. Use the graph of the velocity function below and the notion of Riemann sums to approximate the total displacement of the object from $t = 0$ to $t = 9$ seconds. Use the right endpoint method in your approximation and 9 rectangles. (7 points)



$\approx \frac{1}{2}+4+7+\frac{9}{4}+\frac{9}{4}+\frac{11}{4}+\frac{7}{4}+\frac{3}{4}+0=$

$21.25 = \frac{85}{4}$

9. Use the limit definition of the definite integral to find the area under the curve $y = 2x + 4$ on the interval $[-4,2]$. Sketch the graph of the area. (12 points)

$$\Delta x = \frac{2+4}{n} = \frac{6}{n}$$

$$x_i = -4 + \frac{6i}{n}$$

$$f(x_i) = 2\left(-4+\frac{6i}{n}\right)+4 = -8+\frac{12i}{n}+4 = -4+\frac{12i}{n}$$

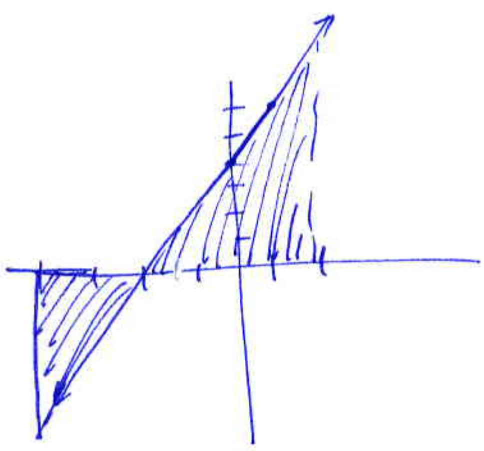
$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(-4+\frac{12i}{n}\right)\frac{6}{n} = \sum_{i=1}^{n}\left(-\frac{24}{n}+\frac{72i}{n^2}\right)$$

$$= \sum_{i=1}^{n} -\frac{24}{n} + \frac{72}{n^2}\sum_{i=1}^{n} i$$

$$-\frac{24}{\cancel{n}}\cdot\cancel{n} + \frac{\overset{36}{\cancel{72}}}{n^2}\left(\frac{n(n+1)}{\cancel{2}}\right) =$$

$$-24 + \frac{36n}{\cancel{n}} + \frac{36}{n}$$

$$\lim_{n\to\infty} 12 + \frac{36}{n} = \boxed{12}$$



10. Use the Fundamental Theorem of Calculus and properties of definite integrals to evaluate the following. (6 points each)

a. $\int_0^1 x - \sqrt{x}\,dx$ $= \int_0^1 x - x^{1/2}\,dx =$

$\frac{1}{2}x^2 - \frac{2}{3}x^{3/2}\Big|_0^1 = \frac{1}{2}(1)^2 - \frac{2}{3}(1)^{3/2} - 0 = \frac{1}{2} - \frac{2}{3} = -\frac{1}{6}$

b. $\int_{-2}^2 x^2 - 4\,dx$ $= 2\int_0^2 x^2 - 4\,dx$ (even)

$2\left[\frac{1}{3}x^3 - 4x\Big|_0^2\right] = 2\left[\frac{8}{3} - 8\right] = -\frac{32}{3}$

c. $\int_0^1 10e^{2x}\,dx$ $= 5e^{2x}\Big|_0^1 = 5e^2 - 5$

d. $\int_{-\pi}^{\pi} \sin(x)\,dx$

odd $= 0$

11. Find the average value of the function $f(x) = \frac{1}{x^2+1}$ on the interval $[-1,1]$. (6 points)

$$\frac{1}{1+1}\int_{-1}^{1}\frac{1}{x^2+1}\,dx \qquad \text{even}$$

$$\frac{2}{2}\int_0^1 \frac{1}{x^2+1}\,dx = \tan^{-1}x\Big|_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

12. Integrate.

a. $\int 2x(x^2+1)^4\,dx$

$u = x^2+1$

$du = 2x\,dx$

$$\int u^4\,du = \frac{1}{5}u^5 + C = \frac{1}{5}(x^2+1)^5 + C$$

b. $\int \frac{8x+6}{2x^2+3x}dx$

$u = 2x^2+3x$

$du = 4x+3\,dx$

$$= \int \frac{2(4x+3)\,dx}{2x^2+3x} = \int \frac{2\,du}{u} = 2\ln|u| + C =$$

$$2\ln|2x^2+3x| + C$$

c. $\int \tan x\,dx$ [Hint: rewrite in terms sine and cosine functions.]

$$\int \frac{\sin x}{\cos x}dx$$

$u = \cos x$

$du = -\sin x\,dx$

$$\int \frac{-1}{u}du = -\ln|u| + C =$$

$$-\ln|\cos x| + C$$

13. Use change of variables to find the integral of $\int x\sqrt[3]{2x+1}dx$.

$$u = \sqrt[3]{2x+1}$$

$$u^3 = 2x+1$$

$$\frac{u^3-1}{2} = x$$

$$\frac{3u^2 du}{2} = dx$$

$$\int \frac{u^3-1}{2} \cdot u \cdot \frac{3u^2}{2} du =$$

$$\frac{1}{4}\int (u^3-1)(3u^3)\, du =$$

$$\frac{1}{4}\int 3u^6 - 3u^3\, du =$$

$$\frac{1}{4}\left[\frac{3}{7}u^7 - \frac{3}{4}u^4\right| + C$$

$$\frac{3}{28}(2x+1)^{7/3} - \frac{3}{16}(2x+1)^{4/3} + C$$