Math 2568, Exam #2, Part 1, Fall 2014 Name KEY

Instructions: On this portion of the exam, you may **NOT** use a calculator. Show all work. Answers must be supported by work to receive full credit.

1. Compute the determinant by the cofactor method. (10 points)

$$\begin{vmatrix} 0 & -3 & 1 & 0 \\ 2 & -1 & 2 & -1 \\ -3 & 3 & -4 & 1 \\ -1 & 0 & 0 & -5 \end{vmatrix} \leftarrow$$

$$\underset{=(1)}{(-1)(-1)} \begin{vmatrix} -3 & 1 & 0 \\ -1 & 2 & -1 \\ 3 & -4 & 1 \end{vmatrix} - 5 \begin{vmatrix} 0 & -3 & 1 \\ 2 & -1 & 2 \\ -3 & 3 & -4 \end{vmatrix} =$$

$$\underset{=(1)}{(-1)(-1)} \begin{vmatrix} -3 & 1 \\ 3 & -4 \end{vmatrix} + 1 \begin{vmatrix} -3 & 1 \\ -1 & 2 \end{vmatrix} - 5 \left[(-1)(-3) \begin{vmatrix} 2 & 2 \\ -3 & -4 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ -3 & 3 \end{vmatrix} \right] =$$

$$(12-3)+(-6+1)-5(3(-8+6)+1(6-3)) =$$

$$9-5-5[3(-2)+3] = 4-5(-6+3) = 4-5(-3) =$$

$$4+15=19$$

2. Compute the determinant by using row operations. (7 points)

$$\begin{vmatrix} 3 & 5 & 4 & 6 \\ -2 & 1 & 0 & 7 \\ -5 & 4 & 7 & 2 \\ 8 & -3 & 1 & 1 \end{vmatrix}$$

(1) $-6R_4+R_1 \to R_1$

(1) $-7R_4+R_2 \to R_2$

(1) $-2R_4+R_3 \to R_3$

$$\begin{array}{rrrr} -48 & 18 & -6 & -6 \\ 3 & 5 & 4 & 6 \\ \hline -45 & 23 & -2 & 0 \end{array} \qquad \begin{array}{rrrr} -56 & 21 & -7 & -7 \\ -2 & 1 & 0 & 7 \\ \hline -58 & 22 & -7 & 0 \end{array}$$

$$\begin{array}{rrrr} -16 & 6 & -2 & -2 \\ -5 & 4 & 7 & 2 \\ \hline -21 & 10 & 5 & 0 \end{array}$$

$-\frac{1}{2}R_1 \to R_1$ $(-1/2)$ **

$$\begin{vmatrix} -45 & 23 & -2 & 0 \\ -58 & 22 & -7 & 0 \\ -21 & 10 & 5 & 0 \\ 8 & -3 & 1 & 1 \end{vmatrix} \to \begin{vmatrix} -45 & 23 & -2 \\ -58 & 22 & -7 \\ -21 & 10 & 5 \end{vmatrix} \to \begin{bmatrix} +22.5 & -11.5 & 1 \\ -58 & 22 & -7 \\ -21 & 10 & 5 \end{bmatrix}$$

$7R_1+R_2 \to R_2$ $-5R_1+R_3 \to R_3$

$$\begin{array}{rrr} 315/2 & -161/2 & 7 \\ -58 & 22 & -7 \\ \hline 199/2 & -117/2 & 0 \end{array} \qquad \begin{array}{rrr} -225/2 & +115/2 & -5 \\ -21 & 10 & 5 \\ \hline -267/2 & 135/2 & 0 \end{array}$$

$$\Rightarrow \begin{vmatrix} +45/2 & -23/2 & 1 \\ 199/2 & -117/2 & 0 \\ -267/2 & 135/2 & 0 \end{vmatrix} = \begin{vmatrix} 199/2 & -117/2 \\ -267/2 & 135/2 \end{vmatrix} = -\frac{2187}{2}$$

$$\left(-\frac{2187}{2}\right)(-2) = 2187$$

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$$\begin{vmatrix} 3 & -1 & 4 & 1 \\ -2 & 1 & 0 & 3 \\ -5 & 4 & 2 & 2 \\ 4 & -3 & 1 & 1 \end{vmatrix} \rightarrow \begin{vmatrix} -13 & 11 & 0 & -3 \\ -2 & 1 & 0 & 3 \\ -13 & 10 & 0 & 0 \\ 4 & -3 & 1 & 1 \end{vmatrix} \rightarrow$$

$-2R_4 + R_3 \rightarrow R_3$ $\quad -8 \quad 6 \quad -2 \quad -2$

$-4R_4 + R_1 \rightarrow R_1$ $\quad -16 \quad 12 \quad -4 \quad -4$

$$-1\begin{vmatrix} -13 & 11 & -3 \\ -2 & 1 & 3 \\ -13 & 10 & 0 \end{vmatrix} \rightarrow -1\begin{vmatrix} 0 & 1 & -3 \\ -15 & 12 & 0 \\ -13 & 10 & 0 \end{vmatrix} \rightarrow$$

$-R_4 + R_1 \rightarrow R_1$ $\quad 13 \quad -10 \quad 0$

$R_1 + R_2 \rightarrow R_2$

$$(-1)(-3)\begin{vmatrix} -15 & 12 \\ -13 & 10 \end{vmatrix} = 3[-150 + 156] = 3(6) = 18$$

3. Given that A and B are $n \times n$ matrices with det A = -4 and det B = 5, find the following. (3 points each)

a) det (BA) $5(-4) = -20$

b) det (A^{-1}) $\frac{1}{-4} = -\frac{1}{4}$

c) det (AB^k) $-4 \cdot (5)^k$

d) det ($-2B^3$) $(-2)^n (5)^3 = 125 \cdot (-2)^n$

e) det (kB) $(k)^n \cdot 5$

f) det ($A^{-1}BA$) $= -\frac{1}{4}(5)(-4) = 5$

4. Suppose that $\det(C) = -20$. Find the determinant of the matrix after the following row operations. (6 points)

$$-2R_1 + R_2 \to R_2, R_1 \leftrightarrow R_3, 2R_2 - 3R_4 \to R_4, \frac{1}{5}R_4 \to R_4$$

(1) (-1) (-3) $(\frac{1}{5})$

$(-1)(-20)(-3)(\frac{1}{5}) = (-4)(-3) = -12$

5. Perform the following matrix operations given $A = \begin{bmatrix} 5 & 3 \\ 4 & -2 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 4 \\ 2 & 1 & -2 \end{bmatrix}, C = \begin{bmatrix} 5 & -5 \\ 2 & 3 \\ 0 & -1 \end{bmatrix}$,

$D = \begin{bmatrix} 4 & 3 & 0 \\ 2 & -1 & 2 \\ 1 & -2 & 1 \end{bmatrix}$. If the operation is not defined, say so. (5 points each)

a. $B^T - 3C$

$$\begin{bmatrix} 2 & 2 \\ 0 & 1 \\ 4 & -2 \end{bmatrix} - 3\begin{bmatrix} 5 & -5 \\ 2 & 3 \\ 0 & -1 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 2 \\ 0 & 1 \\ 4 & -2 \end{bmatrix} + \begin{bmatrix} -15 & 15 \\ -6 & -9 \\ 0 & 3 \end{bmatrix} =$$

$$\begin{bmatrix} -13 & 17 \\ -6 & -8 \\ 4 & 1 \end{bmatrix}$$

c. BC

$$\begin{bmatrix} 2 & 0 & 4 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 5 & -5 \\ 2 & 3 \\ 0 & -1 \end{bmatrix} =$$

$$\begin{bmatrix} 10+0+0 & -10+0-4 \\ 10+2+0 & -10+3+2 \end{bmatrix} =$$

$$\begin{bmatrix} 10 & -14 \\ 12 & -5 \end{bmatrix}$$

b. AC

$$\begin{bmatrix} 5 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 5 & -5 \\ 2 & 3 \\ 0 & -1 \end{bmatrix}$$

2×2 (3×2)

not defined

d. $-11A^{-1}+A$

$$A^{-1} = \frac{1}{-10-12}\begin{bmatrix} -2 & -3 \\ -4 & 5 \end{bmatrix} \Rightarrow$$

$$-11\left(\frac{-1}{22}\begin{bmatrix} -2 & -3 \\ -4 & 5 \end{bmatrix}\right) = \begin{bmatrix} -1 & -3/2 \\ -2 & 5/2 \end{bmatrix}$$

$-11A^{-1}+A =$

$$\begin{bmatrix} -1 & -3/2 \\ -2 & 5/2 \end{bmatrix} + \begin{bmatrix} 5 & 3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 3/2 \\ 2 & 1/2 \end{bmatrix}$$

6. List 12 properties of the Invertible Matrix Theorem. (12 points)

(yours may vary)

A is invertible

A^T is invertible

$\det A \neq 0$

$A\vec{x}=0$ has only trivial solution

$T: A\vec{x} \to b$ is one-to-one

$T: A\vec{x} \to b$ is onto.

$\exists$ a matrix $C \ni CA = I$

$\exists$ a matrix $D \ni AD = I$

A reduces to $n\times n$ identity

A has n pivots

$\text{Nul } A = \{\vec{0}\}$

$\dim \text{Col } A = n$

etc.

7. Determine if each statement is True or False. (1 points each)

a. T F Det(-1A) is always equal to Det(A). Only if n is even

b. T F If the determinant of a matrix is equal to zero, then the columns of the matrix are linearly dependent.

c. T F The determinant of an upper triangular matrix is the product of the entries on the diagonal.

d. T F The product of two matrices A and B is defined in the order AB if A is a mxn matrix and B is an mxp matrix. $n \neq m$

e. T F If A and B are $m \times n$ matrices, then both $\det(AB^T)$ and $\det(A^TB)$ are defined.

f. T F $(AB)^T = A^TB^T$.

g. T F A matrix given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has a unique solution if $ad - bc = 0$.

h. T F If the A matrix is nxn and has n pivots, then the matrix is invertible.

i. T F The coordinates of a position in space are the same regardless of the basis.

j. T F The determinant of $5\begin{vmatrix} a & r & x \\ b & s & y \\ c & t & z \end{vmatrix}$ is the same as the determinant of $\begin{vmatrix} a-3b & r-3s & x-3y \\ b-2c & s-2t & y-2z \\ 5c & 5t & 5z \end{vmatrix}$.

k. T F The inverse of $\begin{bmatrix} 5 & 0 & -1 \\ 2 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}$ is $\frac{1}{33}\begin{bmatrix} 5 & 3 & 1 \\ -6 & 3 & 12 \\ -8 & 15 & 5 \end{bmatrix}$. Check by multiplying

l. T F All systems solved using Cramer's Rule have a unique solution.
nonunique solutions can't use

m. T F If A and P are square matrices then $\det(PAP^{-1}) = \det(A)$.

n. T F If the dimension of the column space of a square nxn matrix A is n, then the matrix A is invertible.

o. T F It's possible to represent elementary row operations as a linear transformation matrix.

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Instructions: On this portion of the exam, you *may* use a calculator to perform elementary matrix operations. Support your answers with work (reproduce the reduced matrices from your calculator) or other justification for full credit.

1. Prove that the set of polynomials $\{t^3+t^2, t^3-t^2, t+1, t-1\}$ is a basis for P_3. Then find the representation of the vector $\vec{p}(t) = 4t^3 - 2t^2 + 3t - 5$ in the basis. Write any change-of-basis matrices P_B employed. (10 points)

$$P_B = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \Rightarrow \text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_B^{-1} = \begin{bmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \end{bmatrix} \qquad P_B^{-1}\begin{bmatrix} -5 \\ 3 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 4 \end{bmatrix}_B$$

thus $(t^3+t^2) + 3(t^3-t^2) - (t+1) + 4(t-1) = 4t^3 - 2t^2 + 3t - 5$.

2. Given the vector $[\vec{x}]_B = \begin{bmatrix} 5 & 7 \\ 8 & -11 \end{bmatrix}$, find the vector $\vec{x}$ in the standard basis given the basis $\mathscr{B}$=$\left\{\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 7 \\ 3 & 5 \end{bmatrix}\right\}$. You will need to create an isomorphism from $M_{2\times2} \rightarrow R^4$, and then solve the problem in R^4 then convert back and check that the change of basis holds. (15 points)

$$x_B = \begin{bmatrix} 5 \\ 7 \\ 8 \\ -11 \end{bmatrix} \qquad \vec{b}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \vec{b}_4 = \begin{bmatrix} 2 \\ 7 \\ 3 \\ 5 \end{bmatrix}$$

$$P_B = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & 2 & 1 & 7 \\ 1 & 1 & 0 & 3 \\ 2 & 1 & 1 & 5 \end{bmatrix} \vec{x} \begin{bmatrix} 5 \\ 7 \\ 8 \\ -11 \end{bmatrix} \qquad P_B[\vec{x}]_B = \begin{bmatrix} -2 \\ -40 \\ -21 \\ -30 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -40 \\ -21 & -30 \end{bmatrix}$$

$$5\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} + 8\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - 11\begin{bmatrix} 2 & 7 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -40 \\ -21 & -30 \end{bmatrix}$$

3. Find the solutions to the system $\begin{cases} 2x_1 + \ x_2 + \ x_3 = 4 \\ -x_1 + \qquad 2x_3 = 2 \\ 3x_1 + x_2 + 3x_3 = -2 \end{cases}$ using Cramer's Rule. (8 points)

$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix}$ $\det A = 4$

$\text{adj} A_1 = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & 1 & 3 \end{bmatrix}$ $\det \text{adj} A_1 = -16$

$\text{adj} A_2 = \begin{bmatrix} 2 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & -2 & 3 \end{bmatrix}$ $\det \text{adj} A_2 = 52$

$\text{adj} A_3 = \begin{bmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & -2 \end{bmatrix}$ $\det \text{adj} A_3 = -4$

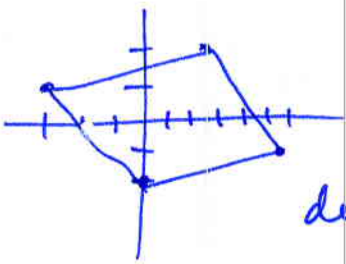
$x_1 = \frac{-16}{4} = -4$

$x_2 = \frac{52}{4} = 13$

$x_3 = \frac{-4}{4} = -1$

$\vec{x} = \begin{bmatrix} -4 \\ 13 \\ -1 \end{bmatrix}$

4. Find the area of the parallelogram with the vertices $(0,-2), (6,-1), (-3,1), (3,2)$. (5 points)



$6-0=6$
$-1+2=1$ $\langle 6,1 \rangle$

$-3-0=-3$
$1+2=3$ $\langle -3,3 \rangle$

$\det \begin{bmatrix} 6 & -3 \\ 1 & 3 \end{bmatrix} = 18+3 = 21$

5. Given the bases B and C, and a vector in one of the bases, find the representation of the vector in the other basis. Verify that both vectors are equivalent to the same vector in the standard basis. (12 points)

$$\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}, \mathscr{C} = \left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix} \right\}, [\vec{x}]_B = \begin{bmatrix} 5 \\ 2 \\ -3 \\ 10 \end{bmatrix}_B$$

$P_B = \begin{bmatrix} 1 & 5 & 2 & 0 \\ -2 & 1 & 2 & 0 \\ 2 & 2 & 1 & 3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ $P_C = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 3 & 1 & 0 & 4 \end{bmatrix}$ $P_C^{-1} = \begin{bmatrix} -4/17 & 11/17 & 8/17 & 1/17 \\ 8/17 & -5/17 & -16/17 & -2/17 \\ 8/17 & -5/17 & 1/17 & -2/17 \\ 1/17 & -7/17 & -2/17 & 4/17 \end{bmatrix}$

$P_B [\vec{x}]_B = P_C [\vec{x}]_C$

$P_C^{-1} P_B [\vec{x}]_B = \underset{C \leftarrow B}{P} [\vec{x}]_B = \begin{bmatrix} -11/17 & 8/17 & 22/17 & 25/17 \\ -12/17 & 1/17 & -16/17 & -50/17 \\ 22/17 & 35/17 & 7/17 & 1/17 \\ 7/17 & -2/17 & -14/17 & -2/17 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ -3 \\ 10 \end{bmatrix}$

$= \begin{bmatrix} 145/17 \\ -528/17 \\ 169/17 \\ 53/17 \end{bmatrix} = [\vec{x}]_C$

6. Find the volume of the parallelepiped bounded by the vertices $(0,0,0), (1,4,0), (-2,-5,2), (-1,2,-1)$. (5 points)

$$\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \\ 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{vmatrix} 1 & -2 & -1 \\ 4 & -5 & 2 \\ 0 & 2 & -1 \end{vmatrix} = 1 \begin{vmatrix} -5 & 2 \\ 2 & -1 \end{vmatrix} + 2 \begin{vmatrix} 4 & 2 \\ 0 & -1 \end{vmatrix} - 1 \begin{vmatrix} 4 & -5 \\ 0 & 2 \end{vmatrix}$$

$$= 1(5-4) + 2(-4-0) - 1(8+0) =$$

$$1 - 8 - 8 = -15$$

$$|-15| = 15$$