Math 2568, Exam #3 – Part I, Fall 2014 Name KEY

Instructions: Show all work. Use exact answers unless specifically asked to round. Justify answers will work or you may receive no credit. You may **not** use a calculator on this portion of the exam.

1. Consider the stochastic Markov chain matrix given by the matrix $A = \begin{bmatrix} .65 & .15 \\ .35 & .85 \end{bmatrix}$. Calculate the equilibrium vector of the system. (6 points)

$\begin{bmatrix} .65-1 & .15 \\ .35 & .85-1 \end{bmatrix} = \begin{bmatrix} -.35 & .15 \\ .35 & -.15 \end{bmatrix}$ $\quad .35x_1 = .15x_2$, $x_2 = x_2$ $\Rightarrow [\vec{x}] = \begin{bmatrix} 3/7 \\ 1 \end{bmatrix}$

$\frac{3}{7} + \frac{7}{7} = \frac{10}{7}$ $\quad \begin{bmatrix} 3/7 \\ 1 \end{bmatrix} \begin{matrix} 7/10 \\ 7/10 \end{matrix} = \begin{bmatrix} 3/10 \\ 7/10 \end{bmatrix}$ or $\begin{bmatrix} .3 \\ .7 \end{bmatrix}$

2. Find the eigenvalues and eigenvectors of the matrices below. Be sure to clearly indicate the characteristic equation, and which eigenvalues and eigenvectors go together. (12 points)

   a. $A = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$

$(5-\lambda)(3-\lambda) - 14 = 15 - 8\lambda + \lambda^2 - 14 = \lambda^2 - 8\lambda + 1 = 0$

$\frac{8 \pm \sqrt{64 - 4(1)(1)}}{2} = \frac{8 \pm \sqrt{60}}{2} = \frac{8 \pm 2\sqrt{15}}{2} = 4 \pm \sqrt{15}$

real.

$\lambda_1 = 4 + \sqrt{15}$
$\lambda_2 = 4 - \sqrt{15}$

λ_1:

$\begin{bmatrix} 5-4-\sqrt{15} & 7 \\ 2 & 3-4-\sqrt{15} \end{bmatrix} = \begin{bmatrix} 1-\sqrt{15} & 7 \\ 2 & -1-\sqrt{15} \end{bmatrix}$ $\quad 2x_1 = (1+\sqrt{15})x_2$

$x_1 = \frac{1+\sqrt{15}}{2} x_2$
$x_2 = x_2$

$\vec{v_1} = \begin{bmatrix} 1+\sqrt{15} \\ 2 \end{bmatrix}$

$\vec{v_2} = \begin{bmatrix} 1-\sqrt{15} \\ 2 \end{bmatrix}$

b. $B = \begin{bmatrix} 5 & 1 \\ -8 & 1 \end{bmatrix}$

$(5-\lambda)(1-\lambda)+8 = 5-6\lambda+\lambda^2+8 = \lambda^2-6\lambda+13=0$

$\lambda = \dfrac{6 \pm \sqrt{36-52}}{2} = \dfrac{6 \pm \sqrt{-16}}{2} = \dfrac{6 \pm 4i}{2} = 3 \pm 2i$ $\quad \lambda_1 = 3+2i$, $\lambda_2 = 3-2i$

λ_1: $\begin{bmatrix} 5-3-2i & 1 \\ -8 & 1-3-2i \end{bmatrix} = \begin{bmatrix} 2-2i & 1 \\ -8 & -2-2i \end{bmatrix}$

$-8x_1 = (2+2i)x_2$

$x_1 = \dfrac{-1-i}{4} x_2$

$x_2 = x_2$

$\vec{v_1} = \begin{bmatrix} -1-i \\ 4 \end{bmatrix}$

$\vec{v_2} = \begin{bmatrix} -1+i \\ 4 \end{bmatrix}$

3. For the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$, with eigenvalues $\lambda_1 = 3, \lambda_2 = 2$ (4) and eigenvectors $\vec{v_1} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$, find a similarity transformation matrix P so that A can be diagonalized. Clearly state both P and D. (7 points)

$P = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 3 \\ 1 & 0 & 0 \end{bmatrix}$ $\qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$x_1 = \frac{1}{3} x_2 + 3x_3$

$x_2 = x_2$

$x_3 = x_3$

$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} x_3$ for $\lambda = 3$

$(3-\lambda)\begin{bmatrix} 4-\lambda & 9 \\ 0 & 3-\lambda \end{bmatrix} = (3-\lambda)\underset{\uparrow}{(4-\lambda)}(3-\lambda) = 0$

$\lambda = 4$

$\lambda = 3$ (repeated)

4. Given the vectors $\vec{u} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix}$ find the following.

a. $\vec{v} \cdot \vec{u}$ (2 points)

$(-2)(3) + (6)(-1) + (3)(0) = -6 - 6 + 0 = -12$

b. $\|\vec{v}\|$. (2 points)

$\sqrt{4 + 36 + 9} = \sqrt{49} = 7$

c. A unit vector in the direction of $\vec{u}$. (2 points)

$\|u\| = \sqrt{9 + (-1)^2} = \sqrt{10}$

$\hat{u} = \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{10} \\ 0 \end{bmatrix}$

d. Find the distance between $\vec{u}$ and $\vec{v}$. (3 points)

$\vec{u} - \vec{v} = \begin{bmatrix} 3-(-2) \\ -1-6 \\ 0-3 \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ -3 \end{bmatrix}$ $\sqrt{25 + 49 + 9} = \sqrt{83}$

e. Are $\vec{u}$ and $\vec{v}$ orthogonal? Why or why not? (2 points)

no, since $\vec{v} \cdot \vec{u} \neq 0$

f. Find $\left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$ (3 points)

$\frac{-12}{49} \begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 24/49 \\ -72/49 \\ -36/49 \end{bmatrix}$

5. Determine if each statement is True or False. (1 point each)

a. T F Every eigenvalue has only one corresponding eigenvector.

b. T F An nxn matrix will always have exactly n real eigenvalues.

c. T F If A and B are row equivalent, then their null spaces are the same.

d. T F $P_{C\leftarrow B} = P_C^{-1}P_B$ $P_C[x]_C = P_B[x]_B$

e. T F A linearly independent set that spans the space in a subspace H is a basis for H.

f. T F If the steady-state vector for a stochastic matrix is unique then the Markov Chain matrix has no absorbing states and has communication between all available states.

g. T F A matrix is invertible if and only if 0 is an eigenvalue of A.

h. T F The eigenvalues of a matrix are always on its main diagonal.

i. T F The eigenspace of an nxn matrix with n distinct real eigenvalues always form a basis for R^n.

j. T F A trajectory of a dynamical system is a set of ordered vectors $\overrightarrow{x_k}$ that tracks the population values of a system over time.

k. T F The elementary row operations of A do not change its eigenvalues.

l. T F If A is invertible, then A is diagonalizable.

m. T F The real eigenvalues of a discrete dynamical system either both attract to the origin or both repel from the origin.

n. T F AB is always the same as BA.

o. T F An inner product space is a vector space with a specific inner product defined on it.

p. T F The vectors in Col A are orthogonal to vectors in Nul A. different size spaces

q. T F If $||\vec{u}||^2 + ||\vec{v}||^2 = ||\vec{u}+\vec{v}||^2$ then $\vec{u}$ and $\vec{v}$ are orthogonal.

Math 2568, Exam #3 – Part II, Fall 2014 Name KEY

Instructions: Show all work. Use exact answers unless specifically asked to round. Justify answers will work or you may receive no credit. You may use a calculator on this portion of the exam.

1. a. For the matrix $B = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$, with eigenvalues $\lambda = 2 \pm i$, with eigenvectors $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \pm 1 \\ 0 \end{bmatrix} i$. Find one similarity transformation P that will transform B=PCP^{-1}, where C is a scaled rotation matrix. State both P and C. (6 points)

$2-i$

$a=2 \quad b=1 \qquad \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$

$P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

b. Use the C matrix from part a, and find the scaling factor and then calculate the angle of rotation of the matrix. Round your angle to 3 decimal places in radians, or to the nearest whole degree. (8 points)

$\sqrt{2^2+1^2} = \sqrt{5}$

$C = \sqrt{5} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

$\cos^{-1}(2/\sqrt{5}) = .4636$ radians

$26.565 \sim 27°$

2. Consider the discrete dynamical system given by the matrix $A = \begin{bmatrix} 1.4 & .5 \\ -1.7 & 1.2 \end{bmatrix}$.

a. Determine the behaviour of the origin for this system: is it a repeller, an attractor or a saddle point? (7 points)

$$(1.4-\lambda)(1.2-\lambda)+1.7(.5)=0$$

$$1.68-2.6\lambda+\lambda^2+.85=0$$

$$\lambda^2-2.6\lambda+2.53=0$$

$$\lambda=\frac{2.6\pm\sqrt{6.76-4(2.53)}}{2}=\frac{2.6\pm i\sqrt{3.36}}{2}\approx 1.3\pm .9165i$$

repeller since

$1.3^2+.9165^2>1$

b. Given the initial condition of the population as $x_0 = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$, find 10 points of the trajectory for the system and list them here. (5 points)

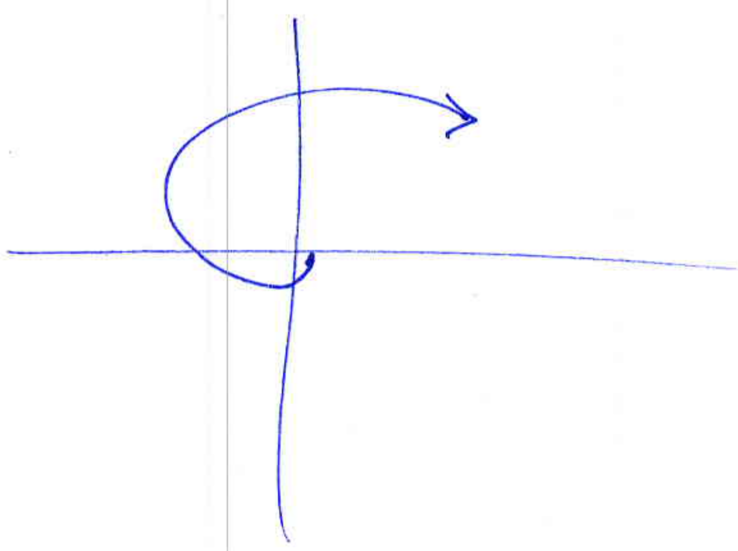
$$\begin{bmatrix} 11.5 \\ 2.3 \end{bmatrix}, \begin{bmatrix} 17.25 \\ -16.79 \end{bmatrix}, \begin{bmatrix} 15.755 \\ -49.473 \end{bmatrix}, \begin{bmatrix} -2.6795 \\ -86.1511 \end{bmatrix}, \begin{bmatrix} -46.83 \\ -98.83 \end{bmatrix},$$

$$\begin{bmatrix} -114.97 \\ -38.99 \end{bmatrix}, \begin{bmatrix} -180.45 \\ 148.67 \end{bmatrix}, \begin{bmatrix} -178.30 \\ 485.17 \end{bmatrix}, \begin{bmatrix} -7.03 \\ 885.3 \end{bmatrix}, \begin{bmatrix} 432.81 \\ 1074.33 \end{bmatrix}$$

c. Plot the points on a graph together with the eigenvectors of the system. Make sure your graph is big enough to clearly read it. Connect the trajectory with a curve and an arrow indicating the flow of time. (8 points)

Spirals outward

no real eigenvalues to plot



3. Solve the differential equation $\vec{x}' = \begin{bmatrix} -7 & 10 \\ -4 & 5 \end{bmatrix} \vec{x}$.

a. State the general solution. (6 points)

$$(-7-\lambda)(5-\lambda)+40=0$$

$$-35+2\lambda+\lambda^2+40=0$$

$$\lambda^2+2\lambda+5=0$$

$$\lambda = \frac{-2\pm\sqrt{4-20}}{2} = \frac{-2\pm 4i}{2} = -1\pm 2i$$

$$\begin{bmatrix} -7+1-2i & 10 \\ -4 & 5+1-2i \end{bmatrix} =$$

$$\begin{bmatrix} -6-2i & 10 \\ -4 & 6-2i \end{bmatrix}$$

$$4x_1 = (6-2i)x_2$$
$$x_2 = x_2$$

$$x_1 = \frac{3-i}{2}x_2$$
$$x_2 = x_2$$

$$\begin{bmatrix} 3-i \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3-i \\ 2 \end{bmatrix} e^{-t}(\cos 2t + i\sin 2t) = e^{-t}\begin{bmatrix} 3\cos 2t + 3i\sin 2t - i\cos 2t + \sin 2t \\ 2\cos 2t + 2i\sin 2t \end{bmatrix}$$

$$\vec{x} = c_1\begin{bmatrix} 3\cos 2t + \sin 2t \\ 2\cos 2t \end{bmatrix} e^{-t} + c_2\begin{bmatrix} 3\sin 2t - \cos 2t \\ 2\sin 2t \end{bmatrix} e^{-t}$$

b. Determine the behaviour of the origin for this system: is it a repeller, an attractor or a saddle point? (7 points)

attractor since real part of λ is negative

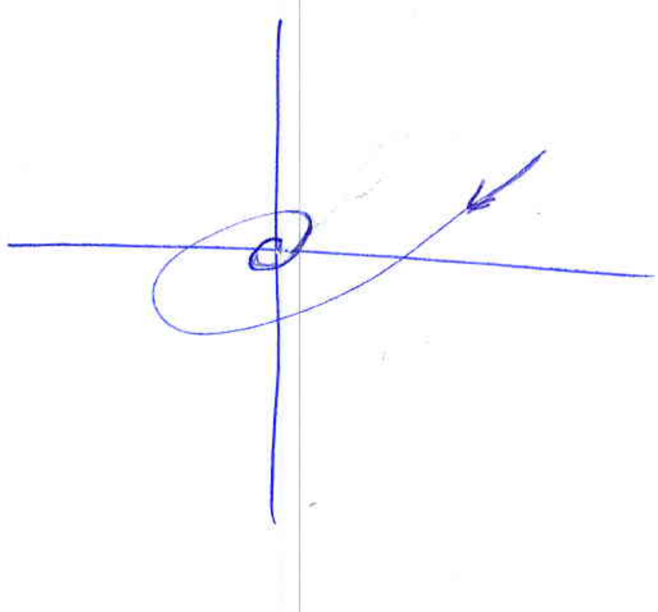
c. Plot the eigenvectors on a graph and plot several sample trajectories of the system. (8 points)

into origin

plot for $c_1 = 10 = c_2$

$$x(t) = (30\cos 2t + 10\sin 2t)e^{-t} + (30\sin 2t - 10\cos 2t)e^{-t}$$
$$= (20\cos 2t + 40\sin 2t)e^{-t}$$

$$y(t) = (20\cos 2t + 20\sin 2t)e^{-t}$$



4. Determine if the functions $f(x) = 4 + x$ and $g(x) = 5 - 4x^2$ are orthogonal under the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$. (8 points)

$$\int_{-1}^{1} (4+x)(5-4x^2)\,dx = \int_{-1}^{1} 20 - 16x^2 \underbrace{+ 5x - 4x^3}_{\substack{\text{odd}\\ =0}} dx$$

$$\int_{-1}^{1} 20 - 16x^2\,dx = 2\int_0^1 20 - 16x^2\,dx = 2\left[20x - \frac{16}{3}x^3\right]_0^1 = 2\left[20 - \frac{16}{3} - 0\right]$$

$$= \frac{88}{3}$$

they are not orthogonal

5. Given the function $f(t) = 3t + t^2$, find a function in P_2 orthogonal to $f(t)$ under the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)dt = 0$. (9 points)

$$g(t) = a + bt + ct^2$$

$$\int_0^1 (3t+t^2)(a+bt+ct^2)dt = \int_0^1 = 3at + 3bt^2 + 3ct^3 + at^2 + bt^3 + ct^4 dt$$

$$= \frac{3a}{2}t^2 + bt^3 + \frac{3c}{4}t^4 + \frac{a}{3}t^3 + \frac{b}{4}t^4 + \frac{c}{5}t^5 \Big|_0^1 =$$

$$\frac{3a}{2} + b + \frac{3c}{4} + \frac{a}{3} + \frac{b}{4} + \frac{c}{5} - 0 = 0$$

$$\left(\frac{3a}{2} + \frac{a}{3}\right) + \left(b + \frac{b}{4}\right) + \left(\frac{3c}{4} + \frac{c}{5}\right) = 0$$

$$\boxed{g(t) = -15 + 22t}$$

$$\left(\frac{9a+2a}{6}\right) + \left(\frac{5b}{4}\right) + \left(\frac{15c+4c}{20}\right) = 0$$

$$\left(\frac{11a}{6}\right) + \left(\frac{5b}{4}\right) + \frac{19c}{20} = 0 \quad * 60$$

$$110a + 75b + 57c = 0$$

let $c = 0$ Then $110a + 75b = 0 \Rightarrow a = \frac{-75}{110}b = \frac{-15}{22}b$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -15 \\ 22 \\ 0 \end{bmatrix}$

6. Answer each of the equations below as completely as possible. (5 points each)

   a. How does one determine the dimension of a vector space (or subspace)?

One looks at the number of elements required for the basis for the space. If no more than 3 vectors can be independent and span the space, the dimension of the space is 3. if an infinite # are required, then the dimension is infinite.

   b. Explain why the origin acts as an attractor when the $|\lambda| < 1$ for a discrete dynamical system, but $\lambda < 0$ is needed for an attractor in an ODE system.

$(e^{\lambda}) < 1$ when $\lambda < 0$

and $(e^{\lambda}) > 1$ when $\lambda > 0$

   c. Explain why the similarity transformation that diagonalizes A, also diagonalizes e^A.

$e^A = \sum_{i=0}^{n} \frac{A^n}{n!}$ So if we diagonalized $A = PDP^{-1}$ or $D = P^{-1}AP$

$e^{D} = \sum_{i=0}^{n} \frac{(P^{-1}AP)^n}{n!}$ or $\sum_{i=1}^{n} \frac{P^{-1}A^n P}{n!}$ and so we

can factor out P and P^{-1} from each term

$e^D = P^{-1} \sum_{i=0}^{n} \frac{A^n}{n!} P$ Thus

$e^A = P \sum_{i=0}^{n} \frac{D^n}{n!} P^{-1}$