MAT 192, Exam #2, Fall 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Verify that $y = \frac{\ln x}{x}$ is a solution to the differential equation $x^2y' + xy = 1$. (5 points)

$x^{-1}\ln x$

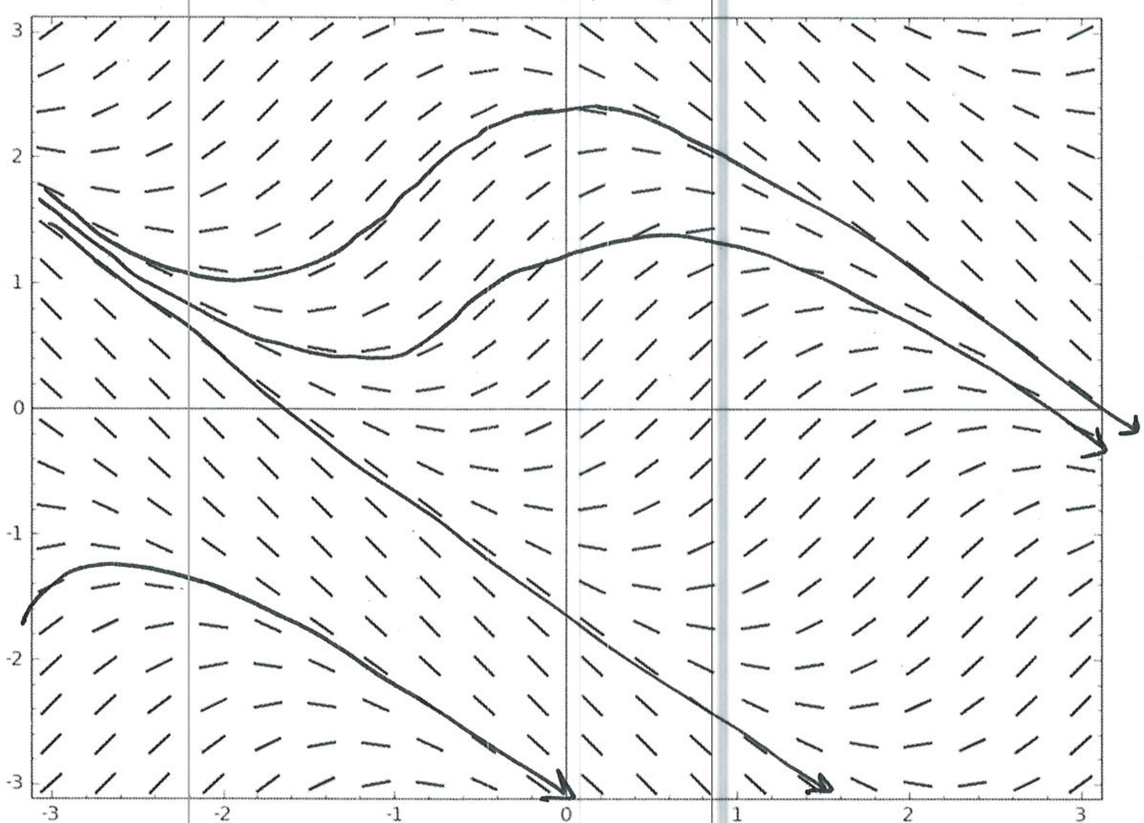
$y' = -x^{-2}\ln x + x^{-1}x^{-1}$

$= -x^{-2}\ln x + x^{-2}$

$x^2(-x^{-2}\ln x + x^{-2}) + x\left(\frac{\ln x}{x}\right) =$

$-\cancel{\ln x} + 1 + \cancel{\ln x} = 1$ ✓

2. Use the graph of the direction/slope field below to plot 4 trajectories with different behaviours, forward and backward, from an initial position. (6 points)



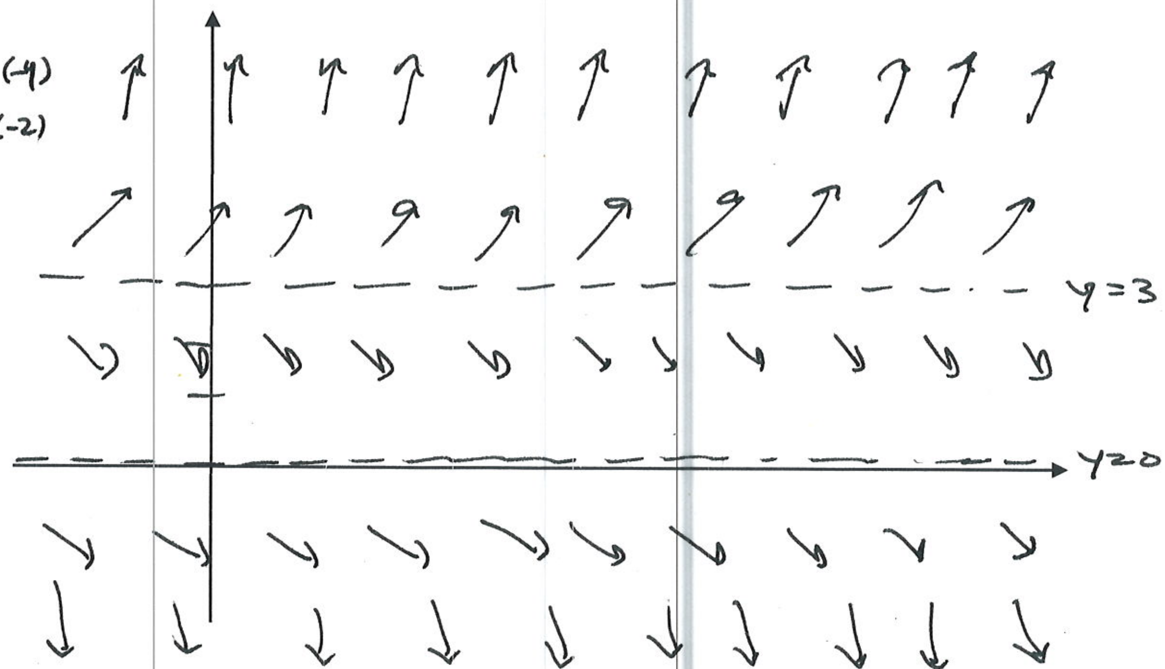
3. Draw a direction field for $y' = y^2(y-3)$ on the graph below. Are the equilibria stable, unstable, or semi-stable. (6 points)

$y=-1$

$(1)(-1-3) = (-4)$

$(1)(1-3) = (1)(-2)$

$(4)^2(4-3) = 16$



4. Find the solution to $y' = \frac{xy\sin x}{y+1}, y(0) = 1$. (8 points)

$\frac{y+1}{y}\,dy = x\sin x\,dx$

$\int 1 + \frac{1}{y}\,dy = \int x\sin x\,dx$

$y + \ln y = -x\cos x + \sin x + C$

$1 + \cancel{\ln(1)}^{0} = -0\cancel{\cos(0)} + \cancel{\sin(0)}^{0} + C$

$C = 1$

$$\boxed{y + \ln y = -x\cos x + \sin x + 1}$$

	u	dv
+	x	sin x
−	1	−cos x
	0	−sin x

5. The solution to the differential equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ has the form $P(t) = \frac{M}{1+Ae^{-kt}}$, $A = \frac{(M-P_0)}{P_0}$. Suppose a population grows according to a logistic model with carrying capacity 6000 and $k = 0.0015/yr$. Use the information above to write the differential equation, and the solution to the system if the initial population is 1000. What is predicted population after 50 years? (8 points)

$M = 6000$

$k = .0015$

$P_0 = 1000$

$A = \frac{6000-1000}{1000} = \frac{5000}{1000} = 5$

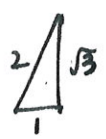
$P(t) = \frac{6000}{1+5e^{-.0015t}}$

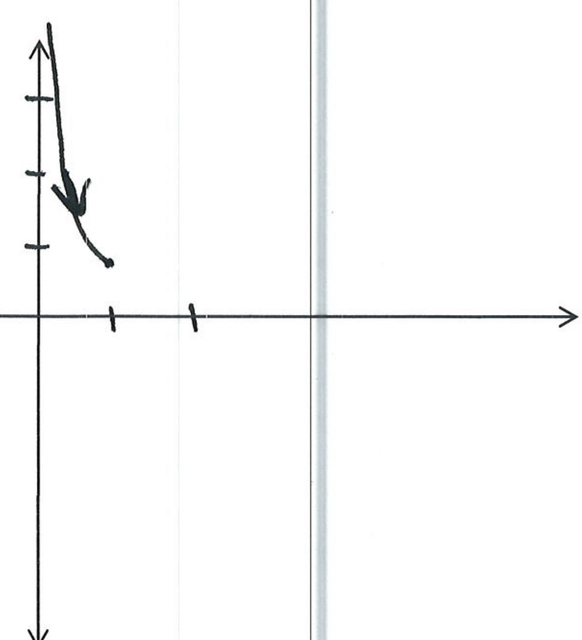
$P(50) = 1064.07$

$\approx \boxed{1064}$

6. Sketch the graph $x = \sin t, y = \csc t$ on the interval $\left(0, \frac{\pi}{2}\right)$. Eliminate the parameter and write the equation in Cartesian coordinates. On your graph, be sure to draw an arrow in the direction of increasing t. (7 points)

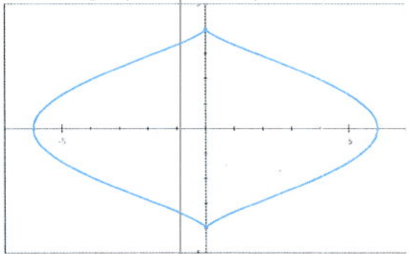
t	x	y
0	0	UND
$\pi/4$	$1/\sqrt{2}$	$\sqrt{2}$
$\pi/2$	1	1
$\pi/6$	$1/2$	2
$\pi/3$	$\sqrt{3}/2$	$2/\sqrt{3}$





$x = \sin t \quad y = \frac{1}{\sin t} \Rightarrow \boxed{y = \frac{1}{x}}$

7. Match the graphs to the parametric equations that generated them. (2 points each)

ii
a.


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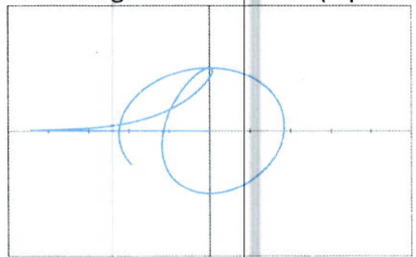
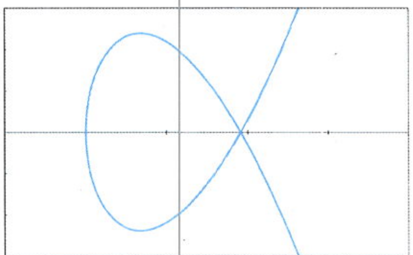

i

v
b.


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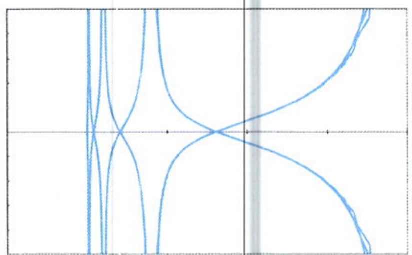
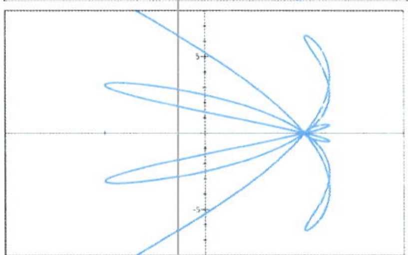
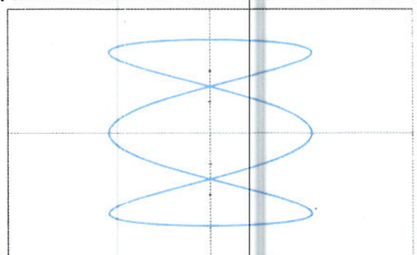

iii

vi
c.


f.

iv.

i. $x = \ln t \cos t\,, y = \sin t$ D
ii. $x = 6\sin^3 t\,, y = 4\cos t$ A
iii. $x = \sec t\,, y = \tan(8t)$ E
iv. $x = \sin(3t)\,, y = 3\cos t$ F
v. $x = \cosh(t)\,, y = (t^2 - 3)\sinh(t)$ B
vi. $x = \sin^2 t + \cos t\,, y = t\cos t$ C

8. Find the equation of the tangent line to the graph $x = t - t^{-1}, y = 1 + t^2$ at $t = 1$. (5 points)

$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$

$\frac{dx}{dt} = 1 + \frac{1}{t^2}$ $\quad \frac{dy}{dt} = 2t$

$\frac{dy}{dx}(1) = \frac{2(1)}{1 + \frac{1}{1^2}} = \frac{2}{2} = 1$

$y(1) = 2$
$x(1) = 0$

$y - 2 = 1(x - 0)$

$\boxed{y = x + 2}$

9. Determine where the curve $x = e^t, y = te^t$ is concave up or concave down. (5 points)

$$\frac{dy}{dt} = e^t + te^t \qquad \frac{dy}{dx} = \frac{e^t + te^t}{e^t} = \frac{\cancel{e^t}(1+t)}{\cancel{e^t}} = 1+t$$

$$\frac{dx}{dt} = e^t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}[1+t]}{e^t} = \frac{1}{e^t} = e^{-t} > 0$$

Concave up $(-\infty, \infty)$ Concave down $\emptyset$

10. Find the length of arc of the curve $y = \frac{1}{2}x^2 - \frac{1}{2}\ln x$ on $[1,2]$. (5 points)

$$\int_1^2 \sqrt{1 + \left(x - \tfrac{1}{2x}\right)^2}\, dx$$

$$\left(x - \tfrac{1}{2x}\right)\left(x - \tfrac{1}{2x}\right) =$$

$$x^2 - \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{4x^2}$$

$$x^2 - 1 + \tfrac{1}{4x^2}$$

$$\int_1^2 \sqrt{x^2 + \tfrac{1}{4x^2}}\, dx = \int_1^2 \sqrt{\frac{4x^4+1}{4x^2}}\, dx$$

$$\int_1^2 \frac{\sqrt{4x^4+1}}{2x}\, dx \approx 1.54568$$

11. Set up the integral to find the length of arc of the curve $x = t^4 - 2t^3 - 2t^2, y = t^3 - 1$ on the interval $[0,2]$. You do not need to integrate. (4 points)

$$\frac{dy}{dt} = 3t^2 \qquad \frac{dx}{dt} = 4t^3 - 6t^2 - 4t$$

$$(4t^3 - 6t^2 - 4t)(4t^3 - 6t^2 - 4t)$$

$$16t^6 - 24t^5 - 16t^4 - 24t^5 + 36t^4$$

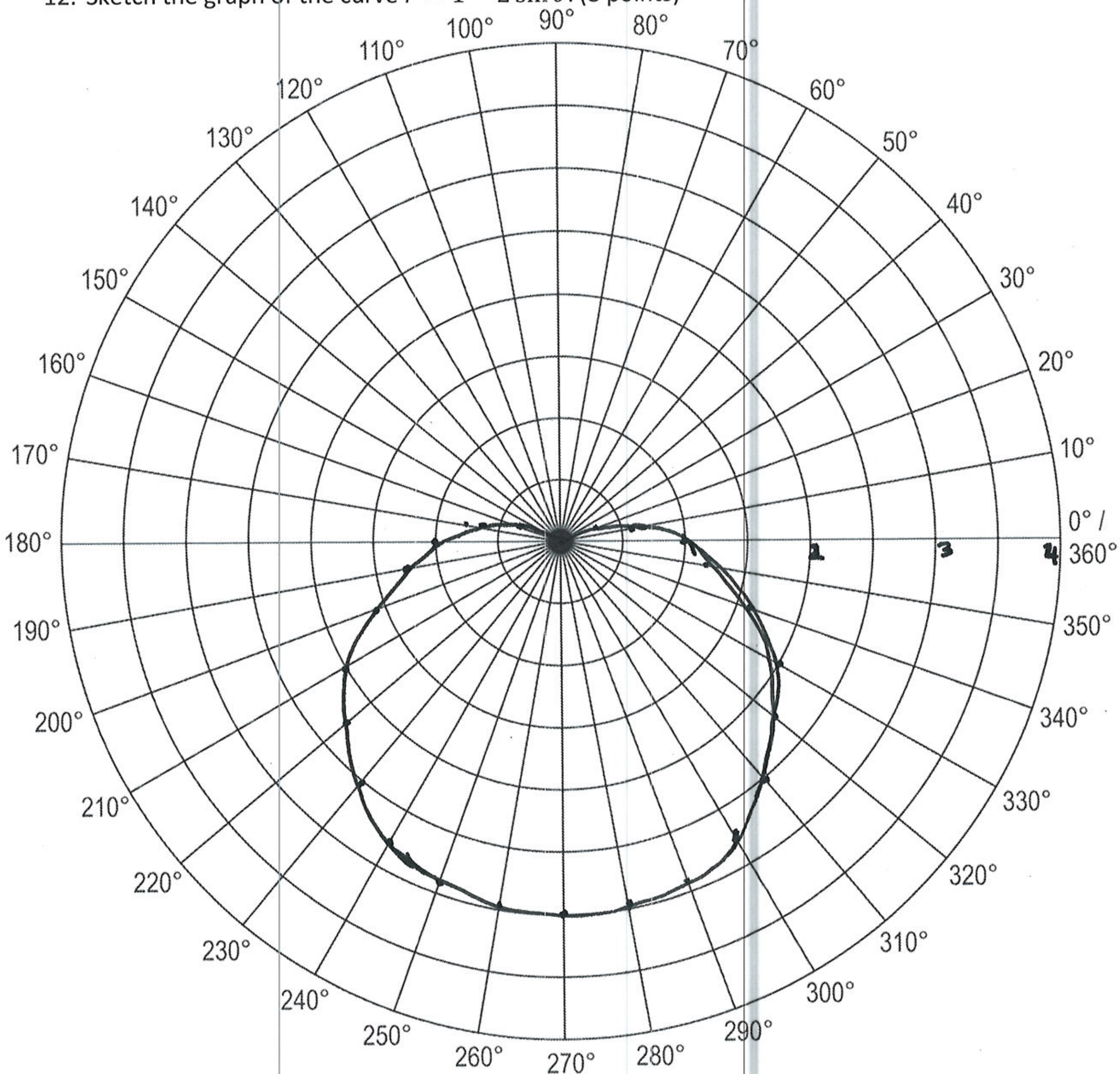
$$+ 24t^3 - 16t^4 + 24t^3 + 16t^2$$

$$s = \int_0^2 \sqrt{(3t^2)^2 + (4t^3 - 6t^2 - 4t)^2}\, dt$$

$$\int_0^2 \sqrt{9t^4 + 16t^6 - 48t^5 + 4t^4 + 48t^3 + 16t^2}\, dt$$

$$\int_0^2 \sqrt{16t^6 - 48t^5 + 13t^4 + 48t^3 + 16t^2}\, dt = \boxed{\int_0^2 t\sqrt{16t^4 - 48t^3 + 13t^2 + 48t + 16}\, dt}$$

12. Sketch the graph of the curve $r = 1 - 2\sin\theta$. (8 points)



13. Rewrite $(x^2 + y^2)^3 = 4x^2y^2$ in polar coordinates. Solve for r. (4 points)

$(r^2)^3 = 4r^2\cos^2\theta \, r^2\sin^2\theta$

$r^6 = 4r^4\cos^2\theta \sin^2\theta$

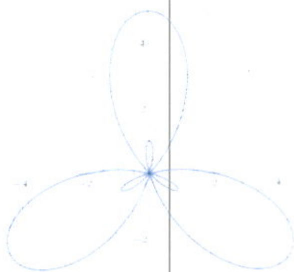
$r^2 = 4\cos^2\theta \sin^2\theta$

$r = 4\cos\theta \sin\theta$

$r = 2\sin 2\theta$

14. Match the equation to the graph of the polar curve. (2 points each)

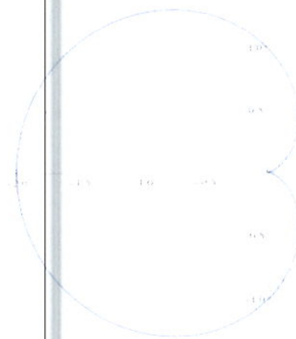
i



a.

V.

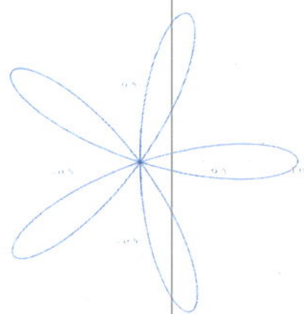
d.



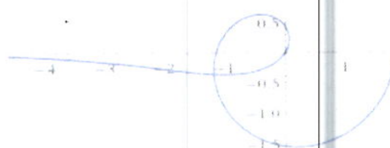
vi

g.

ii

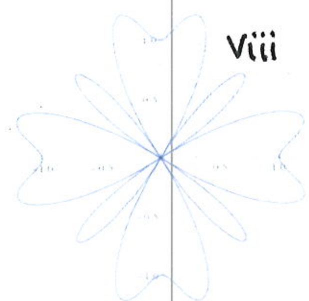


b.



iii

e.

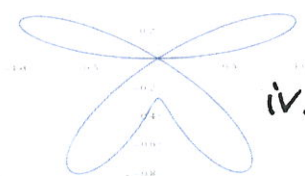


Viii

c.

vii

f.



iv.

h.

i. $r = 2 - 3\sin(3\theta)$ A
ii. $r = \cos(5\theta)$ B
iii. $r = \ln\theta$ E
iv. $r = \sin(6\sin\theta)$ H
v. $r = 4 + 2\sec\theta$ D
vi. $r = 1 - \cos\theta$ G
vii. $r = e^{\frac{\theta}{10}}$ F
viii. $r = \sin^2(4\theta) + \cos(4\theta)$ C

15. Find the length of arc of the curve $r = 2(1 + \cos\theta)$. (5 points)

$$\frac{dr}{d\theta} = 2(-\sin\theta) = -2\sin\theta$$

$$0 = 2(1+\cos\theta)$$
$$-1 = \cos\theta$$
$$\theta = \text{odd } \pi$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 4(1+\cos\theta)^2 + 4\sin^2\theta$$
$$4(1+2\cos\theta+\cos^2\theta) + 4\sin^2\theta$$
$$4 + 8\cos\theta + \underbrace{4\cos^2\theta + 4\sin^2\theta}_{+4} = 8 + 8\cos\theta$$

$$2\int_0^{\pi} \sqrt{8+8\cos\theta}\, d\theta = 16$$

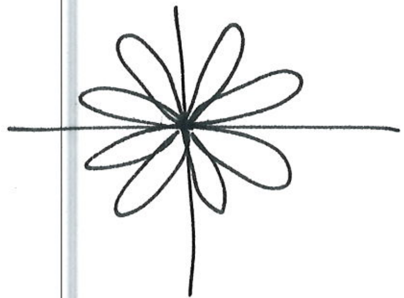
16. Find the area of the region enclosed by one loop of $r = 2\sin 4\theta$. (6 points)

$0 = 2\sin 4\theta$

$0 = \sin 4\theta$

$4\theta = 0, \pi$

$\theta = 0, \pi/4$



$$\frac{1}{2}\int_0^{\pi/4} (2\sin 4\theta)^2 \, d\theta = 2\int_0^{\pi/4} \sin^2 4\theta \, d\theta$$

$$\int_0^{\pi/4} 1 - \cos 8\theta \, d\theta = \theta - \frac{1}{8}\sin 8\theta \Big|_0^{\pi/4} =$$

$$\pi/4 - 0 - 0 + 0 = \boxed{\pi/4}$$

17. Set up the integral to find the area inside both $r = 3\cos\theta$, $r = 1 + \cos\theta$. You do not need to integrate. (5 points)

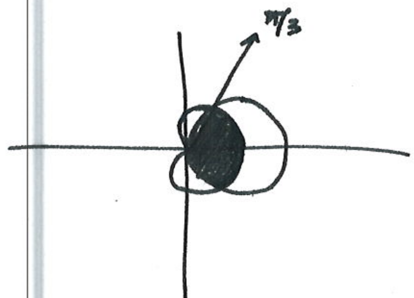
$3\cos\theta = 1 + \cos\theta$

$\frac{2\cos\theta}{2} = \frac{1}{2}$

$\cos\theta = 1/2$

$\theta = \pi/3$

2, $\sqrt{3}$, θ, 1



$$2\left[\int_0^{\pi/3} \frac{1}{2}(1+\cos\theta)^2 \, d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(3\cos\theta)^2 \, d\theta\right] =$$

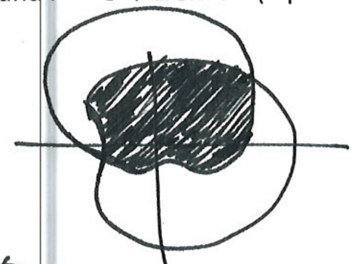
$$\int_0^{\pi/3} 1 + 2\cos\theta + \cos^2\theta \, d\theta + \int_{\pi/3}^{\pi/2} 9\cos^2\theta \, d\theta$$

18. Set up the integral to find the area inside $r = 3 + 2\cos\theta$ and $r = 3 + 2\sin\theta$. (5 points)

$\cancel{3} + 2\cos\theta = \cancel{3} + 2\sin\theta$

$\cos\theta = \sin\theta$

$\pi/4, 5\pi/4$



$$\frac{1}{2}\int_{-3\pi/4}^{\pi/4} (3+2\sin\theta)^2 \, d\theta + \frac{1}{2}\int_{\pi/4}^{5\pi/4} (3+2\cos\theta)^2 \, d\theta$$

19. Determine the type of conic each equation describes. (3 points each)

a. $y + 12x - 2x^2 = 16$

parabola

b. $x^2 + 4y^2 - 18x = 27$

ellipse

c. $4x^2 + 4y^2 + 12x - 20y = 45$

circle

d. $y^2 - 4x^2 - 2y + 16x = 31$

hyperbola

Some useful formulas:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{r(\theta)cos\theta + r'(\theta)\sin\theta}{-r(\theta)\sin\theta + r'(\theta)\cos\theta}$$

$$s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$