192 Homework #10 Key

1a. $\sum_{n=1}^{\infty} \frac{1}{3n^2+2} < \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{n^2}\right) \leftarrow$ converges by p-series $p>1$

Converges by direct comparison

b. $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1} > \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} \leftarrow$ diverges by p-series $p \le 1$

diverges by direct comparison

c. $\sum_{n=1}^{\infty} \frac{3^n}{4^n+5} < \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \leftarrow$ converges by geometric series $\left|\frac{3}{4}\right| < 1$

converges by direct comparison

2a. $\sum_{n=1}^{\infty} \frac{2}{3^n-5}$ compare to $\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges by geometric $\frac{1}{3} < 1$

$$\lim_{n\to\infty} \frac{\frac{2}{3^n-5}}{\left(\frac{1}{3}\right)^n} = \lim_{n\to\infty} \frac{2}{3^n-5}\cdot 3^n = \lim_{n\to\infty} \frac{2\cdot 3^n}{3^n-5} = \lim_{n\to\infty} \frac{2}{1-\frac{5}{3^n}} = 2 \quad (\text{finite}, \neq 0)$$

Converges by limit comparison

b. $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$ compare to $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)

$$\lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sec^2 x}{1} = 1 \quad (\text{finite}, \neq 0)$$

$x = \frac{1}{n}$

diverges by limit comparison

c. $\sum_{n=1}^{\infty} \frac{1}{n(n^2+1)}$ compare to $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by p-series $p>1$

$$\lim_{n\to\infty} \frac{\frac{1}{n^3+n}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3}{n^3+n} = 1$$ converges by limit comparison

3a. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-series $p \le 1$ (or use integral test)

b. $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$ Compare $<$ $\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges by geometric series $\left|\frac{1}{3}\right| < 1$ (or multiply by $\frac{3^{-n}}{3^{-n}}$, use integral test)

direct comparison, converges

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3k. $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$ this may be telescoping, or use integral or comparison test

$\sum_{n=1}^{\infty} \frac{3}{n(n+3)} < 3\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series; so original converges by direct comparison

l. $\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)} = \sum_{k=1}^{\infty} \frac{(2k-1)\cancel{(k+1)}(k-1)}{\cancel{(k+1)}(k^2+4)} = \sum_{k=1}^{\infty} \frac{(2k-1)(k-1)}{(k^2+4)}$

nth term test $\lim_{n\to\infty} \frac{(2k-1)(k-1)}{k^2+4} = 2 \neq 0$ diverges

Can also use integral test (after long division) or comparison w/ $\sum_{k=1}^{\infty} 2$.

m. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n} = \sum_{n=1}^{\infty} \frac{\sqrt[n]{e}}{n}$ compare to $\sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series diverges by p-series

$\lim_{n\to\infty} \frac{\frac{\sqrt[n]{e}}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt[n]{e}}{\cancel{n}} \cdot \frac{\cancel{n}}{1} = 1$ diverges by limit comparison

n. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ compare to $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series

$1/n = x$

$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1$ diverges by comparison

4a. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+1}$ diverges $\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$ by alternating series test

b. $\sum_{n=1}^{\infty} \cos(n\pi) = \sum_{n=1}^{\infty} (-1)^n$ diverges by alternating series test

c. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{\sqrt[3]{n}}$ $\lim_{n\to\infty} \frac{n^{1/2}}{n^{1/3}} = \lim_{n\to\infty} n^{1/6} = \infty$ diverges by alternating series test

d. $\sum_{n=0}^{\infty} (-1)^n e^{-n^2}$ $\lim_{n\to\infty} e^{-n^2} = 0$ converges by AST

$\sum_{n=0}^{\infty} e^{-n^2} < \sum_{n=0}^{\infty} e^{-n}$ converges by direct comparison - converges absolutely

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4e. $\sum_{n=1}^{\infty} (-1)^{n+1} \arctan n$ $\quad \lim_{n\to\infty} \arctan n = \pi/2 \neq 0$ diverges

f. $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$ $\quad \lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = 0$ converges by alternating series test

converges conditionally

$\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n}\right)$ diverges by limit comparison w/ $\pi \sum_{n=1}^{\infty} \frac{1}{n}$ (see 3n)

g. $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n})$ $\quad \lim_{n\to\infty} \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}} = 0$

converges by alternating series.

Converges by telescoping series $-\sum_{n=1}^{\infty} (\sqrt{n} - \sqrt{n+1})$ absolutely

h. $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{(2n+1)!}$ $\quad \lim_{n\to\infty} \frac{3^n}{(2n+1)!} = 0$ converges by ALT

$\lim_{n\to\infty} \frac{3^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{3^n} = \lim_{n\to\infty} \frac{3}{(2n+2)(2n+3)} = 0$ converges absolutely

i. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln(n+1)}{n+1}$ $\quad \lim_{n\to\infty} \frac{\ln(n+1)}{n+1} = 0$ converges by ALT (conditionally)

diverges $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$ by direct comparison w/ $> \sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges

(or use integral test)

j. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sqrt{n}}{n+2}$ $\quad \lim_{n\to\infty} \frac{\sqrt{n}}{n+2} = 0$ converges by ALT (conditionally)

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+2}$ diverges by comparison w/ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-series

k. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$ $\quad \lim_{n\to\infty} \frac{1}{\ln n} = 0$ converges conditionally by ALT

diverges (absolutely) by direct comparison w/ $\frac{1}{n}$

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4d. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ $\lim_{n\to\infty} \frac{1}{n+1} = 0$ Converges by ALT

$\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges by Comparison w/ $\sum_{n=1}^{\infty} \frac{1}{n}$

m. $\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$ $\lim_{n\to\infty} \sqrt[n]{e^2} = 1$ diverges

n. $\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{n!}$ $\lim_{n\to\infty} \frac{n^n}{n!} = \infty$ diverges

o. $\sum_{n=1}^{\infty} (-1)^{n-1} n e^{-n}$ $\lim_{n\to\infty} \frac{n}{e^n} = 0$ converges by ALT

$\lim_{n\to\infty} \left| \frac{(n+1)e^{-(n+1)}}{n e^{-n}} \right| = \frac{1}{e}$ converges absolutely

5a. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} = \frac{1}{\sqrt{e}}$ $\frac{1}{2^n n!} < .001 \Rightarrow 2^n n! > 1000$

$n = 5$ $2^5 5! = 3840$

b. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ $\frac{1}{n^2} < .001 \Rightarrow n^2 > 1000$ $n > 31.62..$

$n = 32$

c. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n 4^n} = \ln\left(\frac{5}{4}\right)$ $\frac{1}{n4^n} < .001 \Rightarrow n4^n > 1000$

$4 \cdot 4^4 = 1024$ $n = 4$

6a. $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \left(\frac{1}{n!}\right)$ $\lim_{n\to\infty} \frac{\left(\frac{3}{4}\right)^{n+1} / (n+1)!}{\left(\frac{3}{4}\right)^n / n!} = \lim_{n\to\infty} \frac{3}{4} \cdot \frac{1}{n+1} = 0 < 1$

converges by ratio test

b. $\sum_{n=0}^{\infty} \frac{3^n}{(n+1)^n}$ $\lim_{n\to\infty} \sqrt[n]{\frac{3^n}{(n+1)^n}} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$ converges by root test

c. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-series

d. $\sum_{n=1}^{\infty} \frac{n}{4^n}$ $\lim_{n\to\infty} \sqrt[n]{\frac{n}{4^n}} = \frac{1}{4} < 1$ Converges by root test

e. $\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$ $\lim_{k\to\infty} \sqrt[k]{k\left(\frac{2}{3}\right)^k} = \frac{2}{3}$ converges by root test

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6f. $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{3}\right)}{n!} < \sum_{n=1}^{\infty} \frac{1}{n!}$ $\lim_{n\to\infty} \frac{n!}{(n+1)!} = \frac{1}{n+1} = 0 < 1$ converges by ratio test

Converges by direct comparison

g. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} \Rightarrow \lim_{n\to\infty} \frac{(2n+2)!}{[(n+1)!]^2} \cdot \frac{(n!)^2}{(2n)!} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)} = 4 > 1$

diverges by ratio test

h. $\sum_{n=1}^{\infty} \frac{2^n n!}{5 \cdot 8 \cdot 11 \cdots (3n+2)}$ $\lim_{n\to\infty} \frac{\cancel{2^{n+1}}(n+1)\cancel{!}}{\cancel{5 \cdot 8 \cdot 11 \cdots (3n+2)}(3n+5)} \cdot \frac{\cancel{5 \cdot 8 \cdot 11 \cdots (3n+2)}}{\cancel{2^n}\,\cancel{n!}} = \frac{2(n+1)}{3n+5} = \frac{2}{3} < 1$

Converges by ratio test

i. $\sum_{n=1}^{\infty} \frac{n^2+1}{n!}$ $\lim_{n\to\infty} \frac{n^2+2n+2}{(n+1)!} \cdot \frac{n!}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$

Converges by ratio test

j. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$ $\lim_{n\to\infty} \left| \frac{2^{4n+4}}{(2n+3)!} \cdot \frac{(2n+1)!}{2^{4n}} \right| = \lim_{n\to\infty} \frac{2^4}{(2n+2)(2n+3)} = 0 < 1$

Converges by ratio test (absolutely)

k. $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$ $\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{(\ln n)^n}\right|} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 < 1$ Converges by root test

l. $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$ $\lim_{n\to\infty} \sqrt[n]{\frac{(n!)^n}{n^{2n}}} = \lim_{n\to\infty} \frac{n!}{n^2} = \infty$ diverges by root test

m. $\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n} < \sum_{n=1}^{\infty} \frac{1}{4^n}$ converges by geometric series test

Converges by direct comparison

n. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$ $\lim_{n\to\infty} \sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

o. $\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$ $\lim_{n\to\infty} \frac{2^{\cancel{n^2}+2n+1}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{\cancel{2^{n^2}}} = \lim_{n\to\infty} \frac{2^{2n+1}}{n+1} = \infty$ diverges by ratio test