192 Homework # 12 Key

1a. $f(x) = e^{-x}$, $n=3$

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(c)$	$(x-c)^n$	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	e^{-x}	1	1	1
1	1	$-e^{-x}$	-1	x	$-x$
2	2	e^{-x}	1	x^2	$\frac{x^2}{2}$
3	6	$-e^{-x}$	-1	x^3	$-\frac{x^3}{6}$

$$P_3(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$$

b. $f(x) = xe^x$, $n=4$

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(c)$	$(x-c)^n$	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	xe^x	0	1	0
1	1	$e^x + xe^x = (x+1)e^x$	1	x	x
2	2	$e^x + (x+1)e^x = (x+2)e^x$	2	x^2	x^2
3	6	$(x+3)e^x$	3	x^3	$\frac{x^3}{2}$
4	24	$(x+4)e^x$	4	x^4	$\frac{x^4}{6}$

$$P_4(x) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6}$$

192 Homework #12 Key

1c. $f(x) = \tan x$, $n=4$

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(c)$	$(x-c)^n$	$\frac{f^{(n)}(c)(x-c)^n}{n!}$
0	1	$\tan x$	0	1	0
1	1	$\sec^2 x$	1	x	x
2	2	$2\sec^2 x \tan x$	0	x^2	0
3	6	$4\sec^2 x \tan^2 x + 2\sec^4 x$	2	x^3	$\frac{x^3}{3}$
4	24	$8\sec^2 x \tan^3 x + 8\tan x \sec^4 x + 8\sec^4 x \tan x$	0	x^4	0

$$P_4(x) = x + \frac{x^3}{3}$$

d. $f(x) = \sin \pi x$, $n=4$

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(c)$	$(x-c)^n$	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	$\sin \pi x$	0	1	0
1	1	$\pi \cos \pi x$	π	x	πx
2	2	$-\pi^2 \sin \pi x$	0	x^2	0
3	6	$-\pi^3 \cos \pi x$	$-\pi^3$	x^3	$\frac{\pi^3 x^3}{6}$
4	24	$\pi^4 \sin \pi x$	0	x^4	0

$$P_4(x) = \pi x + \frac{\pi^3 x^3}{6}$$

192 Homework #12 Key

1e. $f(x) = \frac{x}{x+1}$, $n=5$

$$x+1 \overline{) \, x} \quad \text{quotient } 1, \quad \underline{-x \mp 1} \quad = 1 - \frac{1}{x+1} = 1-(x+1)^{-1}$$

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(c)$	$(x-c)^n$	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	$1-\frac{1}{x+1}$	0	1	0
1	1	$(x+1)^{-2}$	1	x	x
2	2	$-2(x+1)^{-3}$	-2	x^2	$-x^2$
3	6	$6(x+1)^{-4}$	6	x^3	x^3
4	24	$-24(x+1)^{-5}$	-24	x^4	$-x^4$
5	120	$120(x+1)^{-6}$	120	x^5	x^5

$P_5(x) = x - x^2 + x^3 - x^4 + x^5$

2a. $f(x) = \frac{1}{n}$, $n=4$, $c=1$

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(c)$	$(x-c)^n$	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	x^{-1}	1	1	1
1	1	$-x^{-2}$	-1	x	$-x$
2	2	$2x^{-3}$	2	x^2	x^2
3	6	$-6x^{-4}$	-6	x^3	$-x^3$
4	24	$24x^{-5}$	24	x^4	x^4

$P_4(x) = 1 - x + x^2 - x^3 + x^4$

192 Homework #12 Key

2b. $f(x) = x^2 \cos x$, $n = 3$, $c = \pi$

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(c)$	$(x-c)^n$	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	$x^2 \cos x$	$-\pi^2$	1	$-\pi^2$
1	1	$2x\cos x - x^2 \sin x$	-2π	$x-\pi$	$-2\pi(x-\pi)$
2	2	$2\cos x - 4x\sin x - x^2\cos x$	$-2+\pi^2$	$(x-\pi)^2$	$(-2+\pi^2)(x-\pi)^2$
3	6	$-6\sin x - 6x\cos x + x^2 \sin x$	$+6\pi$	$(x-\pi)^3$	$6\pi(x-\pi)^2$

$$P_3(x) = -\pi^2 - 2\pi(x-\pi) + (-2+\pi^2)(x-\pi)^2 + 6\pi(x-\pi)^3$$

2c. $f(x) = x - x^3$, $c = -2$

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(c)$	$(x-c)^n$	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	$x - x^3$	6	1	6
1	1	$1-3x^2$	-11	$x+2$	$-11(x+2)$
2	2	$-6x$	12	$(x+2)^2$	$\frac{12}{2}(x+2)^2$
3	6	-6	-6	$(x+2)^3$	$\frac{-6}{6}(x+2)^3$

$$P_3(x) = -(x+2)^3 + 6(x+2)^2 - 11(x+2) + 6$$

192 Homework #12 Key

3d. $f(x) = \sqrt[3]{x} = x^{1/3}$, $n=3$, $c=8$

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(c)$	$(x-c)^n$	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	$x^{1/3}$	2	1	2
1	1	$\frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$	$\frac{1}{12}$	$x-8$	$\frac{1}{12}(x-8)$
2	2	$-\frac{2}{9}x^{-5/3} = \frac{-2}{9\sqrt[3]{x^5}}$	$-\frac{1}{144}$	$(x-8)^2$	$\frac{-1}{288}(x-8)^2$
3	6	$\frac{10}{27}x^{-8/3} = \frac{10}{27\sqrt[3]{x^8}}$	$\frac{5}{3456}$	$(x-8)^3$	$\frac{5}{20{,}736}(x-8)^3$

$$P_3 = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20{,}736}(x-8)^3$$

3e. $f(x) = x^4 - 3x^2 + 1$, $c=1$

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(1)$	$(x-c)^n$	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	x^4-3x^2+1	-1	1	-1
1	1	$4x^3-6x$	-2	$x-1$	$-2(x-1)$
2	2	$12x^2-6$	6	$(x-1)^2$	$3(x-1)^2$
3	6	$24x$	24	$(x-1)^3$	$4(x-1)^3$
4	24	24	24	$(x-1)^4$	$(x-1)^4$

$$P_4(x) = -1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4$$

192 Homework #12 Key

3. $P_0(x) = 1$

$P_1(x) = 1 - x$

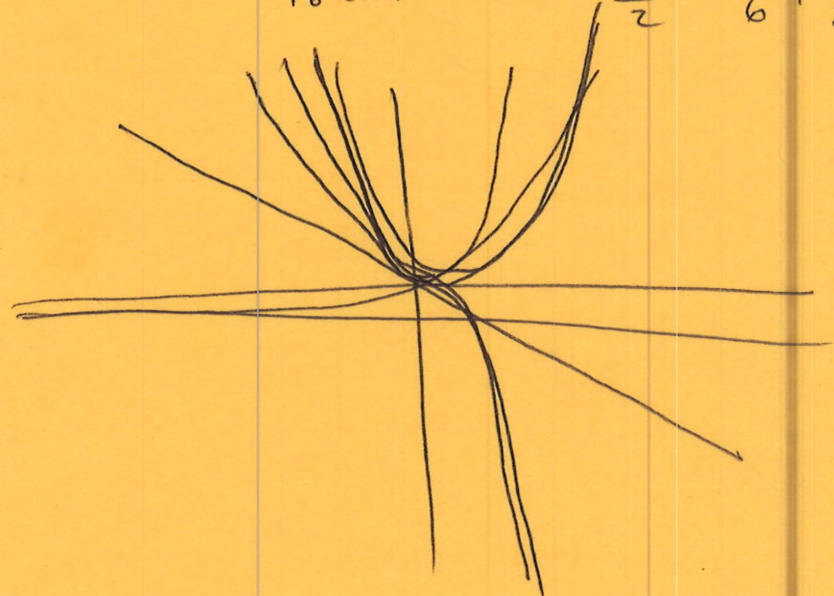
$P_2(x) = 1 - x + \frac{x^2}{2}$

$P_3(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$

$P_4(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$

$P_5(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$

$P_6(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720}$



4a. $f(x) = e^{3x}$ $\quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$= \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

b. $f(x) = \sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \cdots - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} \cdots\right)\right]$

$= \frac{1}{2}\left[\cancel{1} + x + \cancel{\frac{x^2}{2}} + \frac{x^3}{6} + \cancel{\frac{x^4}{24}} \cdots - \cancel{1} + x - \cancel{\frac{x^2}{2}} + \frac{x^3}{6} - \cancel{\frac{x^4}{24}} \cdots\right]$

$= \frac{1}{2}\left[2x + \frac{2x^3}{6} + \frac{2x^5}{120} + \cdots\right]$

$= x + \frac{x^3}{6} + \frac{x^5}{120} \cdots$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

192 Homework #12 Key

4c. $f(x) = 2\sin x^3$ $\quad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$= 2\sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{2(-1)^n x^{6n+3}}{(2n+1)!}$$

d. $f(x) = e^x \cos x$ $\quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$

$$\cos x = \sum_{n=0}^{\infty} \frac{x^{2n}(-1)^n}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots$$

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} \dots\right) =$$

$$1 - \cancel{\frac{x^2}{2}} + \frac{x^4}{24} + x - \cancel{\frac{x^3}{2}} + \cancel{\frac{x^2}{2}} - \frac{x^4}{4} + \cancel{\frac{x^3}{6}} + \frac{x^4}{24} \dots$$

$$1 + x - \frac{x^3}{3} - \frac{x^4}{6}, \dots$$

$$\approx 1 + x - \frac{x^3}{6} - \frac{x^4}{6} + \dots$$

e. $f(x) = x^2 \ln(1 + x^3)$ $\quad \ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$

$$x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+5}}{n+1}$$

f. $f(x) = \ln(x^2+1)$ $\quad \ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}$$

g. $f(x) = \cos 4x,\ c = \pi/2$ $\quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (4(x - \pi/2))^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 16^n (x - \pi/2)^{2n}}{(2n)!}$$

h. $f(x) = \cos^2 x$ $\quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots\right)$

A2 Homework #12 Key

4h. Cont'd $\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots\right)^2=\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots\right)\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots\right)$

$$1-\cancel{\frac{x^2}{2}}+\cancel{\frac{x^4}{24}}-\frac{x^6}{720}-\cancel{\frac{x^2}{2}}-\frac{x^6}{48}+\cancel{\frac{x^4}{24}}-\frac{x^6}{48}-\frac{x^6}{720}+\cdots$$

$$=1-x^2+\frac{x^4}{12}-\frac{x^6}{24}+\cdots$$

i. $f(x)=\frac{\sin x}{1+x}$ $\qquad \sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\begin{array}{r} x-x^2+\frac{5}{6}x^3-\frac{5}{6}x^4+\frac{101}{120}x^5-\frac{101}{120}x^6 \\ 1+x\,\big)\,x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}\cdots \\ -\,x\mp x^2 \\ \hline -x^2-\frac{x^3}{6} \\ +x^2+x^3 \\ \hline \frac{5}{6}x^3 \\ -\frac{5}{6}x^3\mp\frac{5}{6}x^4 \\ \hline -\frac{5}{6}x^4+\frac{x^5}{120} \\ +\frac{5}{6}x^4+\frac{5}{6}x^5 \\ \hline \frac{101}{120}x^5 \\ -\frac{101}{120}x^5\mp\frac{101}{120}x^6 \\ \hline -\frac{101}{120}x^6-\frac{x^7}{5040} \\ +\frac{101}{120}x^6+\frac{101}{120}x^7 \\ \hline \frac{4241}{5040}x^7\ \cdots \end{array}$$

$$=x-x^2+\frac{5}{6}x^3-\frac{5}{6}x^4+\frac{101}{120}x^5-\frac{101}{120}x^6+\frac{4241}{5040}x^7+\cdots$$

192 Homework #12 Key

4j. $f(x) = \arctan(x^3)$ $\quad \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n+1}$$

5a. $\int x \cos x^3 \, dx$ $\quad \cos x = \sum_{n=0}^{\infty} \frac{(-1) x^{2n}}{(2n)!}$

$$x \cos x^3 = x \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!}$$

$$\int x \cos x^3 \, dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{6n+1}}{(2n)!} dx = \sum_{n=0}^{\infty} \frac{(-1) x^{6n+2}}{(2n)!(6n+2)} + C$$

b. $\int \frac{e^x - 1}{x} dx$ $\quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$ $\quad \frac{e^x - 1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$

$$= \int \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} dx = \sum_{n=1}^{\infty} \frac{x^n}{(n!) n} + C$$

c. $\int_0^1 \sin x^4 \, dx$ $\quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n (x^4)^{\overset{8n+4}{2n+1}}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+5}}{(2n+1)!(8n+5)} \Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n (1)^{\cancel{8n+5}}}{(2n+1)!(8n+5)}$$

d. $\int_0^{1/2} x^2 e^{-x^2} dx$ $\quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$ $\quad x^2 e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!}$

$$\int_0^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{n!(2n+3)} \Big|_0^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (1/2)^{2n+3}}{n!(2n+3)}$$

6a. $$\lim_{x \to 0} \frac{x - \ln(1+x)}{x^2} = \lim_{x \to 0} \frac{\cancel{x} - (\cancel{x} - \frac{x^2}{2} + \frac{x^3}{3} - \cdots)}{x^2} = \lim_{n \to \infty} \frac{\frac{x^2}{2} - \frac{x^3}{3} + \cdots}{x^2} = \frac{1}{2}$$

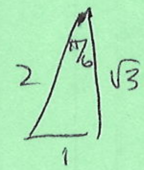
b. $$\lim_{x \to 0} \frac{1 - \cos x}{1 + x - e^x} = \lim_{x \to 0} \frac{\cancel{1} - (\cancel{1} - \frac{x^2}{2} + \frac{x^4}{24} + \cdots)}{\cancel{1} + \cancel{x} - (\cancel{1} + \cancel{x} + \frac{x^2}{2} + \frac{x^3}{6} + \cdots)} = \lim_{x \to 0} \frac{\frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} + \cdots}{-\frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{24} - \cdots} = -1$$

192 Homework #12 Key

7a. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!} = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = e^{-x^4}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3^n}{n\, 5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\left(\frac{3}{5}\right)^n\left(\frac{3}{5}\right) = \frac{3}{5}\ln\left(1+\frac{3}{5}\right)$

c. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{6}\right)^{2n} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$



8a. $f(x) = e^{2x}$, $c = 0$, $a = 1$ $\quad [0,1]$

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = 1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{6} + \frac{16x^4}{24} + \frac{32x^5}{120} + \frac{64x^6}{720} + \frac{128x^7}{5040} + \frac{256x^8}{40{,}320}$$

$$+ \frac{512x^9}{9!} + \frac{1024x^{10}}{10!} + \frac{2048x^{11}}{11!} + \frac{4096x^{12}}{12!}$$

$$R = \frac{2^{12}x^{12}e^2}{12!} \xleftarrow{x=1} = 6.318\times10^{-5} < 10^{-4}$$

b. $f(x) = (8+x)^{1/3}$ $\quad [2, 2.5]$ $\quad c = 2$

$$[10 + (x-2)]^{1/3} = 10^{1/3}\left(1 + \frac{x-2}{10}\right)^{1/3}$$

$$10^{1/3} + 10^{1/3}(1/3)\frac{(x-2)}{10} + \frac{10^{1/3}(1/3)(-2/3)(x-2)^2}{10^2} + \underbrace{\frac{10^{1/3}(1/3)(-2/3)(-5/3)(x-2)^3}{10^3}}_{E = 9.974\times10^{-5}}$$

Remainder