MAT 201, Exam #3, Fall 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Use Stokes' Theorem to set up the integral to evaluate $\int\int_R \vec{\nabla}\times\vec{F}\cdot d\vec{S}$ for $\vec{F}(x,y,z) = x^2z^2\hat{\imath} + y^2z^2\hat{\jmath} + xyz\hat{k}$ where S is the part of the paraboloid $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 4$ oriented upward. (7 points)

$z - x^2 - y^2 = G$
$\nabla G = \langle -2x, -2y, 1 \rangle$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2z^2 & y^2z^2 & xyz \end{vmatrix} = (xz - 2y^2z)\hat{\imath} - (yz - 2x^2z)\hat{\jmath} + (0-0)\hat{k}$$

$$\vec{\nabla}\times\vec{F}\cdot\nabla G = (-2x)(xz-2y^2z) - (-2y)(yz-2x^2z) + 0(1) =$$
$$-2x^2z + \cancel{4xy^2z} + 2y^2z - \cancel{4x^2yz} = (-2x^2+2y^2)(z) =$$
$$(-2x^2+2y^2)(x^2+y^2) = -2x^4 - \cancel{2x^2y^2} + \cancel{2x^2y^2} + 2y^4 = -r^4\cos^4\theta + r^4\sin^4\theta$$
$$r^4\,(\underbrace{\sin^2\theta+\cos^2\theta}_{=1})(\sin^2\theta - \cos^2\theta)$$

$$\int_0^{2\pi}\int_0^2 -2(r^4)(\underbrace{\sin^2\theta - \cos^2\theta}_{-\cos 2\theta})\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 2r^5\cos 2\theta\,dr\,d\theta =$$

$$2\int_0^{2\pi} \frac{r^6}{6}\Big|_0^2 \cos 2\theta\,d\theta = \frac{2}{6}\int_0^{2\pi} 64\cos 2\theta\,d\theta = \frac{128}{6}\cdot\frac{1}{2}\sin 2\theta\Big|_0^{2\pi} = 0$$

2. Set up an integral to find the length of arc on $0 \le t \le 1$ for the curve defined by $\vec{r}(t) = 3t^2\hat{\imath} + 8t^{3/2}\hat{\jmath} + 12t\hat{k}$. [You do not need to evaluate it.] (6 points)

$$r'(t) = 6t\hat{\imath} + 8\cdot\tfrac{3}{2}t^{1/2}\hat{\jmath} + 12\hat{k} = 6t\hat{\imath} + 12\sqrt{t}\,\hat{\jmath} + 12\hat{k}$$
$$\|r'(t)\| = \sqrt{36t^2 + 144t + 144} = 6\sqrt{t^2+4t+4} = 6(t+2)$$

$$\int_0^1 6(t+2)\,dt$$

3. Find the unit tangent vector for $\vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + \ln\cos t\,\hat{k}$. (6 points)

$$r'(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + (-\tan t)\hat{k}$$

$$\|r'(t)\| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t} = \sqrt{1+\tan^2 t} = \sqrt{\sec^2 t} = \sec t = \frac{1}{\cos t}$$

$$T(t) = -\sin t\cos t\,\hat{\imath} + \cos^2 t\,\hat{\jmath} - \sin t\,\hat{k}$$

4. Find the curvature of the function $\vec{r}(t) = t \ln t\,\hat{\imath} + t\hat{\jmath} + e^{-t}\hat{k}$. Evaluate it at $t = 1$. What is the radius of curvature at the same point? (7 points)

$$r'(t) = (\ln t + 1)\hat{\imath} + 1\hat{\jmath} + (-e^{-t})\hat{k}$$

$$r''(t) = \left(\tfrac{1}{t}\right)\hat{\imath} + 0\hat{\jmath} + e^{-t}\hat{k}$$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \ln t + 1 & 1 & -e^{-t} \\ \frac{1}{t} & 0 & e^{-t} \end{vmatrix} = (e^{-t} - 0)\hat{\imath} - \left(e^{-t}\ln t + \frac{e^{-t}}{t}\right)\hat{\jmath} - \frac{1}{t}\hat{k}$$

$$\|\vec{r}' \times \vec{r}''\| = \sqrt{e^{-2t} + \left(e^{-t}\ln t + \frac{e^{-t}}{t}\right)^2 + \frac{1}{t^2}} \quad t=1 \Rightarrow \sqrt{e^{-2} + (e^{-1}(0) + e^{-1})^2 + 1/1} = \sqrt{2e^{-2}+1}$$

$$\|r'(t)\| = \sqrt{(\ln t + 1)^2 + 1^2 + e^{-2t}} \quad \Rightarrow \sqrt{(0+1)^2 + 1 + e^{-2}} = \sqrt{2+e^{-2}}$$

$$K(1) = \frac{\sqrt{2e^{-2}+1}}{(2+e^{-2})^{3/2}} \qquad R = \frac{(2+e^{-2})^{3/2}}{\sqrt{2e^{-2}+1}}$$

5. Find an equation of the tangent plane to the surface $z = x^2 + xy + 3y^2$ at the point (1,1,5). (6 points)

$$F = x^2 + xy + 3y^2 - z$$

$$\nabla F = \langle 2x+y,\ x+6y,\ -1\rangle$$

$$\nabla f(1,1)\ \langle 3,\ 7,\ -1\rangle$$

$$3(x-1) + 7(y-1) - 1(z-5) = 0$$

6. Find an equation for the normal line to the surface $z = x^2 + xy + 3y^2$ at the point (1,1,5). (3 points)

$$\langle 3, 7, -1\rangle$$

$$\vec{n}(t) = (3t+1)\hat{\imath} + (7t+1)\hat{\jmath} + (5-t)\hat{k}$$

7. Find the equation of the tangent plane to the surface $\vec{r}(u,v) = u\sin 2v\,\hat{\imath} + u^2\hat{\jmath} + u\cos 2v\,\hat{k}$ at $(4,16,0)$. (6 points)

$u=4$
$v=\pi/4$

$\vec{r}_u = \sin 2v\,\hat{\imath} + 2u\,\hat{\jmath} + \cos 2v\,\hat{k}$
$\vec{r}_v = 2u\cos 2v\,\hat{\imath} + 0\hat{\jmath} - 2u\sin 2v\,\hat{k}$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sin 2v & 2u & \cos 2v \\ 2u\cos 2v & 0 & -2u\sin 2v \end{vmatrix} = (-4u^2\sin 2v - 0)\hat{\imath} - (-2u\sin^2 2v - 2u\cos^2 2v)\hat{\jmath} + (0 - 4u^2\cos 2v)\hat{k}$$

$-4u^2\sin 2v\,\hat{\imath} - 2u\hat{\jmath} - 4u^2\cos 2v\,\hat{k}$

$\langle -64, -8, 0\rangle \qquad \langle 8, 1, 0\rangle$

$-64(x-4) + (-8)(y-16) + 0(z-0) = 0$

or $\quad 8(x-4) + (y-16) = 0$

8. Use the chain rule to find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$ for $z = e^{x+2y}, x = \frac{s}{t}, y = \frac{t}{s}$. Be sure to write both derivatives only in terms of s and t. [You do not need to simplify.] (6 points)

$z = e^x e^{2y}$

$\frac{\partial z}{\partial x} = e^{x+2y} \qquad \frac{\partial z}{\partial y} = 2e^{x+2y} \quad \Rightarrow \quad \frac{\partial z}{\partial x} = e^{s/t + 2t/s} \qquad \frac{\partial z}{\partial y} = 2e^{s/t + 2t/s}$

$\frac{\partial x}{\partial t} = \frac{-s}{t^2} \qquad \frac{\partial x}{\partial s} = \frac{1}{t}$

$\frac{\partial y}{\partial t} = \frac{1}{s} \qquad \frac{\partial y}{\partial s} = \frac{-t}{s^2}$

$$\frac{\partial z}{\partial t} = \left(e^{s/t + 2t/s}\right)\left(-\frac{s}{t^2}\right) + \left(2e^{s/t + 2t/s}\right)\left(\frac{1}{s}\right)$$

$$\frac{\partial z}{\partial s} = \left(e^{s/t + 2t/s}\right)\left(\frac{1}{t}\right) + \left(2e^{s/t + 2t/s}\right)\left(-\frac{t}{s^2}\right)$$

9. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $yz + x\ln y = z^2$. (6 points)

$F = yz + x\ln y - z^2$

$F_x = \ln y$
$F_y = z + \frac{x}{y}$
$F_z = y - 2z$

$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-\ln y}{y - 2z}$

$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{-(z + \frac{x}{y})}{y - 2z}$

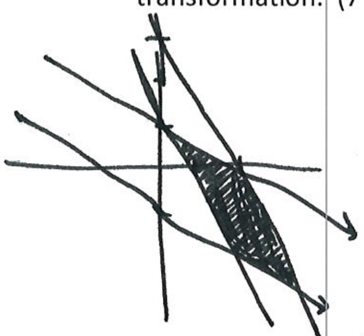
10. Find the Jacobian for the change of variables given by $x = v, y = u(1 + v^2)$. (6 points)

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 0 & 1 \\ 1+v^2 & 2uv \end{vmatrix} = 0 - (1+v^2)$$

$$= -(1+v^2) \text{ or}$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1+v^2$$

11. Sketch the region bounded by $y = 2x - 1, y = 2x + 1, y = 1 - x, y = 3 - x$. Set up a change of variables for the region. Solve for x and y in terms of u and v, and sketch the region after the transformation. (7 points)



$y - 2x = -1$ $\quad u = y - 2x \quad [-1,1]$

$y - 2x = 1$ $\quad v = y + x \quad [1,3]$

$y + x = 1$

$y + x = 3$

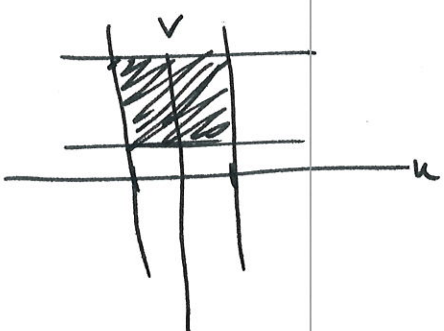
$u - v = -3x \Rightarrow x = -\frac{1}{3}(u-v)$ or $\frac{1}{3}(v-u)$

$v = y + \frac{1}{3}(v-u)$

$v = y + \frac{1}{3}v - \frac{1}{3}u$

$-\frac{1}{3}v$

$\frac{1}{3}u + \frac{2}{3}v = y \Rightarrow y = \frac{1}{3}(u+2v)$



$$\boxed{\begin{aligned} x &= \tfrac{1}{3}(v-u) \\ y &= \tfrac{1}{3}(u+2v) \end{aligned}}$$

12. Find the total differential of the function $R = \alpha\beta^2 \cos\gamma$. Use the value of the function at $(1,2,\pi)$ to estimate the value at $\left(0.9, 2.05, \frac{5\pi}{6}\right)$. (6 points)

$\Delta\alpha = -.1 \quad \Delta\beta = .05 \quad \Delta\gamma = -\pi/6$

$\frac{\partial R}{\partial \alpha} = \beta^2 \cos\gamma \Rightarrow 4\cos\pi = -4$

$\frac{\partial R}{\partial \beta} = 2\alpha\beta\cos\gamma \Rightarrow 4\cos\pi = -4$

$\frac{\partial R}{\partial \gamma} = -\alpha\beta^2 \sin\gamma \Rightarrow 0$

$\Delta R \approx -4(-.1) - 4(.05) + (0)(-\pi/6) = .4 - .2 = .2$

$R(1,2,\pi) = 4\cos\pi = -4 \qquad R(.9, 2.05, 5\pi/6) \approx -4 + .2 = -3.8$

13. Find the directional derivative for the function $f(x,y)=\frac{y^2}{x}$ at the point $P(1,2)$ in the direction of $\vec{v}=2\hat{\imath}+\sqrt{5}\hat{\jmath}$. (5 points)

$\nabla f = \langle -\frac{y^2}{x^2}, \frac{2y}{x} \rangle \Rightarrow \langle -4, 4 \rangle$

$\|\vec{v}\| = \sqrt{4+5} = \sqrt{9} = 3$

$\hat{u} = \frac{2}{3}\hat{\imath} + \frac{\sqrt{5}}{3}\hat{\jmath}$

$\vec{\nabla} f \cdot \hat{u} = -\frac{8}{3} + \frac{4\sqrt{5}}{3} = \frac{4\sqrt{5}-8}{3}$

14. Consider the function $f(x,y)=\frac{1}{3}y^3 - xy + 4y - \frac{1}{2}x^2 + 6x - 11$. Find $\vec{\nabla} f$. Sketch the graph of the gradient field by graphing the curves $f_x = 0$ and $f_y = 0$. Find any critical points and use the direction field to determine if each critical point is a maximum, a minimum or a saddle point (or cannot be determined). Verify your results with the second partials test. (12 points)

$\langle -y-x+6,\ y^2-x+4 \rangle$

$f_x=0 \Rightarrow y=-x+6 \quad f_y=0 \Rightarrow y^2=x-4$

$(-x+6)^2 = x-4$

$(x-6)^2 = x-4$

$x^2-12x+36 = x-4$

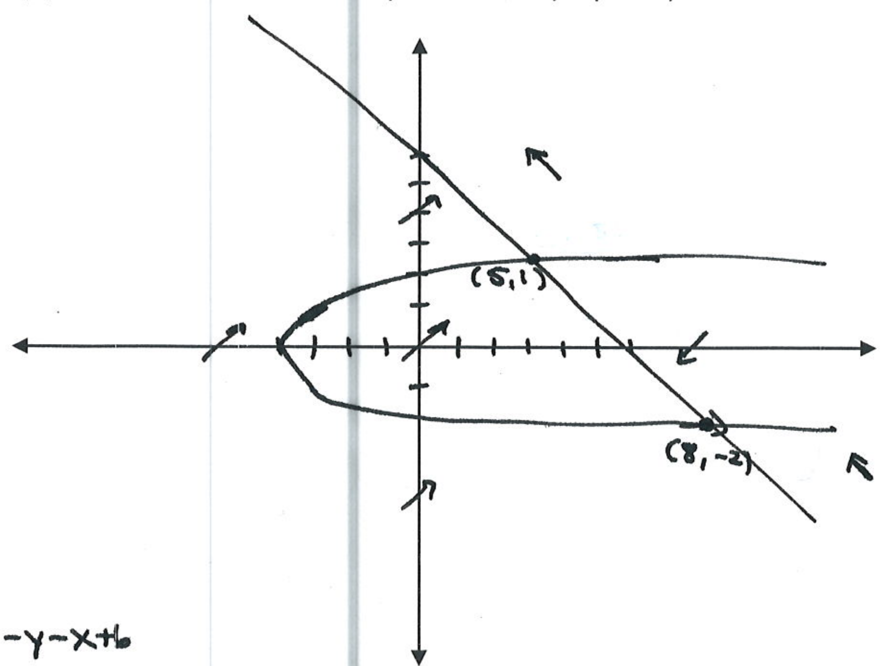
$x^2-13x+40=0$

$(x-8)(x-5)=0$

$x=8, x=5$

$y=-8+6=-2$

$y=-5+6=1$



$\nabla f(0,0) = \langle 6,4 \rangle$

$\nabla f(5,3) = \langle -2,9 \rangle$

$\nabla f(0,4) = \langle 2,20 \rangle$

$\nabla f(8,0) = \langle -2,-4 \rangle$

$\nabla f(-5,0) = \langle 11,9 \rangle$

$\nabla f(0,-5) = \langle 11,29 \rangle$

$\nabla f(10,-2) = \langle -2, 2 \rangle$

$\nabla f(10,-3) = \langle -1,3 \rangle$

$f_x = -y-x+6$

$f_y = y^2-x+4$

$f_{xx} = -1$

$f_{yy} = 2y$

$f_{xy} = -1$

$D(5,1) = (-1)(2)-(-1)^2 = -3$ Saddle point

$D(8,-2) = (-1)(-4)-(-1)^2 = 3$ and $f_{xx}, f_{yy} < 0$ maximum

(5,1) saddle point, (8,-2) maximum.

15. Find the absolute extrema of the function $f(x,y) = 4x + 6y - x^2 - y^2$ on the region $[0,4] \times [0,5]$. Sketch the region. (8 points)

$f_x = 4 - 2x = 0 \quad 4 = 2x \quad x = 2$

$f_y = 6 - 2y = 0 \Rightarrow 6 = 2y \quad y = 3$

$x = 0 \quad f(0,y) = 6y - y^2$

$f'(y) = 6 - 2y$

$x = 4 \quad f(4,y) = \cancel{16} + 6y - \cancel{16} - y^2$

$f'(y) = 6 - 2y$

$y = 0 \quad f(x,0) = 4x - x^2$

$f'(x) = -2x + 4$

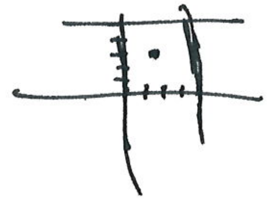
$y = 5 \quad f(x,5) = 4x + 30 - x^2 - 25$

$5 + 4x - x^2$

$f'(x) = 2 - 2x$

+ corner points

$f(2,3) = 13$
$f(0,3) = 9$
$f(4,3) = 9$
$f(2,0) = 4$
$f(2,5) = 9$
$f(0,0) = 0$
$f(4,0) = 0$
$f(4,5) = 5$
$f(0,5) = 5$



check
(2,3)
(0,3)
(4,3)
(2,0)
(2,5)
(0,0)
(4,0)
(4,5)
(0,5)

max at $f(2,3) = 13$

min @ $f(0,0), f(4,0) = 0$

16. Set up an integral to find the surface area of $z = \frac{2}{3}\left(x^{3/2} + y^{3/2}\right)$ over the region $[0,1] \times [0,1]$. [You do not need to evaluate it.] (7 points)

$F = \frac{2}{3}(x^{3/2} + y^{3/2}) - z$

$\nabla F = \langle x^{1/2}, y^{1/2}, -1 \rangle$

$\|\nabla F\| = \sqrt{x + y + 1}$

$$\int_0^1 \int_0^1 \sqrt{x + y + 1}\, dy\, dx$$

17. Set up an integral to find the surface area of the function $\vec{r}(u,v) = u\cos v\,\hat{\imath} + u\sin v\,\hat{\jmath} + v\hat{k}$ for $0 \le u \le 1, 0 \le v \le \pi$. [You do not need to evaluate it.] (7 points)

$\vec{r}_u = \cos v\,\hat{\imath} + \sin v\,\hat{\jmath} + 0\hat{k}$

$\vec{r}_v = -u\sin v\,\hat{\imath} + u\cos v\,\hat{\jmath} + \hat{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = (\sin v)\hat{\imath} - (\cos v)\hat{\jmath} + (u\cos^2 v + u\sin^2 v)\hat{k}$$

$\sin v\,\hat{\imath} - \cos v\,\hat{\jmath} + u\hat{k}$

$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1 + u^2}$

$$\int_0^1 \int_0^\pi \sqrt{1 + u^2}\, dv\, du$$

18. Evaluate the surface integral $\int\int_S (x+y+z)dS$ where S is $\vec{r}(u,v) = (u+v)\hat{\imath} + (u-v)\hat{\jmath} + (1+2u+v)\hat{k}$ for $[0,2] \times [0,1]$. (9 points)

$\vec{r}_u = \hat{\imath} + \hat{\jmath} + 2\hat{k}$

$\vec{r}_v = \hat{\imath} - \hat{\jmath} + \hat{k}$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = (1+2)\hat{\imath} - (1-2)\hat{\jmath} + (-1-1)\hat{k}$

$3\hat{\imath} + \hat{\jmath} - 2\hat{k}$

$ds = \sqrt{9+1+4} = \sqrt{14}$

$x+y+z = (u+v)+(u-v) + (1+2u+v)$

$= 2u+1+2u+v =$

$4u+v+1$

$\int_0^2 \int_0^1 (4u+v+1)\sqrt{14}\, dv\,du$

$\sqrt{14} \int_0^2 \int_0^1 4u+v+1\, dv\,du$

$= \sqrt{14} \int_0^2 4uv + \frac{1}{2}v^2 + v \Big|_0^1 du$

$= \sqrt{14} \int_0^2 4u + \frac{3}{2}\, du =$

$\sqrt{14} \left[2u^2 + \frac{3}{2}u \Big|_0^2 \right. =$

$\sqrt{14}\,[8+3] = 11\sqrt{14}$

19. Use the Divergence Theorem to calculate the flux through the field $\vec{F}(x,y,z) = x^4\hat{\imath} - x^3z^2\hat{\jmath} + 4xy^2z\hat{k}$ for S: the surface bounded by the cylinder $x^2+y^2=1$ and the planes $z = x+2$ and $z=0$. [You do not need to evaluate it.] (7 points)

$\vec{\nabla}\cdot\vec{F} = 4x^3 + (0) + 4xy^2 = 4x^3 + 4xy^2 = 4x(x^2+y^2)$

$4r\cos\theta \cdot r^2 = 4r^3\cos\theta$

$\int_0^{2\pi} \int_0^1 \int_0^{r\cos\theta+2} 4r^4\cos\theta\, dz\,dr\,d\theta$

20. Describe the difference between a sink, a source and an incompressible fluid in the context of the Divergence Theorem. (5 points)

a sink has a negative divergence (more goes in than comes out)
a source has a positive divergence (more comes out than goes in)
an incompressible fluid is a flow which is zero
(the same amount goes in as comes out)

Useful formulas:

$$K = \frac{\left|f''(x)\right|}{\left(1+\left[f'(x)\right]^2\right)^{\frac{3}{2}}}$$

$$K = \frac{\left\|\vec{r}\,'(t)\times\vec{r}\,''(t)\right\|}{\left\|\vec{r}\,'(t)\right\|^3}$$