Mass & Center of Mass

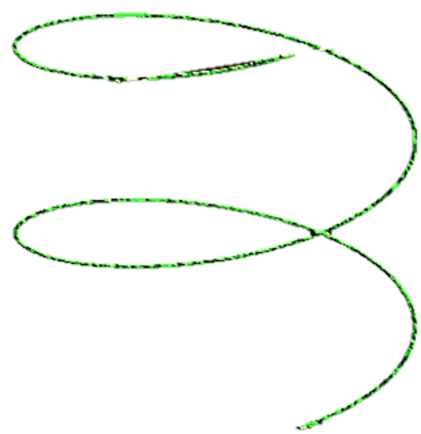
Calculating the total mass of an object requires a density function and a region (or path) the object occupies, and integrating over that region or path. This handout will discuss several scenarios encountered in multivariable calculus including mass of a wire, mass of a lamina in a plane (in rectangular and polar coordinates), mass of a lamina over a surface, mass of a volume (in rectangular, cylindrical and spherical coordinates). From there, we will extend the discussion to calculating the center of mass of a lamina in a plane and center of mass of a volume in various coordinate systems.

1. **Mass of a wire**
   The mass of a wire has a density function typically provided in terms of density per unit length. Consequently, we can integrate this function only one time to obtain the total mass. For these kinds of problems, we will need a line integral. In general, we can describe the total mass as:

$$M = \int_C \rho(x, y, z) ds$$

Where C is a parametric equation that describes how the wire moves through space, $\vec{r}(t)$; ρ is the density function, and $ds = \|\vec{r}'(t)\| dt$.

Example 1. Find the mass of a wire with density function $\rho(x, y, z) = kx^2z$, over the path of a helix $\vec{r}(t) = 4\cos(t)\,\hat{\imath} + 4\sin(t)\,\hat{\jmath} + \frac{1}{2}t$ for two cycles.



First, we find the density function in terms of the path, but substituting $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ into the density function.

$$x(t) = 4\cos(t), z(t) = \frac{1}{2}t$$
$$\rho(t) = k[4\cos(t)]^2\left(\frac{1}{2}t\right) = 8kt\cos^2(t)$$

Next, we need to calculate ds.

$$\vec{r}'(t) = -4\sin(t)\,\hat{\imath} + 4\cos(t)\,\hat{\jmath} + \frac{1}{2}\hat{k}$$
$$\|\vec{r}'(t)\| = \sqrt{16\cos^2(t) + 16\sin^2(t) + \left(\frac{1}{2}\right)^2} = \sqrt{16 + \frac{1}{4}} = \sqrt{\frac{65}{4}} = \frac{\sqrt{65}}{2}$$

Two cycles of this helix is $0 \le t \le 4\pi$.

Putting these elements together we obtain our integral $M = \int_0^{4\pi} 8kt\cos^2(t) \cdot \frac{\sqrt{65}}{2} dt$.

Completing the integration: $4\sqrt{65}k \int_0^{4\pi} t \cdot \frac{1}{2}(1 + \cos(2t))dt = 2\sqrt{65}k \int_0^{4\pi} t + t\cos(2t)\, dt$

The second piece needs to be completed by parts, with $u = t, dv = \cos(2t)$. The result is

$$2\sqrt{65}k\left[\frac{1}{2}t^2 + \frac{1}{2}t\sin(2t) - \frac{1}{4}\cos(2t)\right]_0^{4\pi} = \sqrt{65}k\left[(4\pi)^2 + 0 - \frac{1}{2}(1) - 0 - 0 + \frac{1}{2}(1)\right] =$$

$$16\pi^2\sqrt{65}k$$

If $k = 1$, this is approximately 1273.14 mass units.

Practice Problems.

Find the mass of the wire over the given path, and with the given density function.

a. $\rho(x, y, z) = kyz, \vec{r}(t) = t\hat{\imath} + t^2\hat{\jmath} + 4t\hat{k}, 0 \le t \le 2$
b. $\rho(x, y, z) = k|xy|, \vec{r}(t) = \sqrt{t}\hat{\imath} + (t-3)\hat{\jmath} + 4\hat{k}, 0 \le t \le 5$

2. Mass of a lamina in a plane

i. Rectangular Coordinates

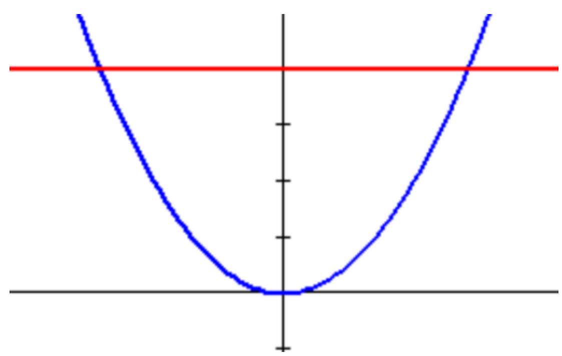
We will next look at calculating the pass of a lamina (a thin sheet) in a plane. We will need the boundaries of the lamina R, and the density function $\rho(x, y)$ in units of mass per unit area. While this type of problem can be done with one integral if the density function is constant, it requires two integrals for variable density.

$$M = \int\int_R \rho(x, y)\, dA$$

Example 2. Find the mass of the lamina bounded by $y = x^2, y = 4$, with density function $\rho(x, y) = ky$.

The limits of integration are the boundaries of the area of the lamina, and we integrate the density function. Thus we obtain the integral

$$M = \int_{-2}^{2}\int_{x^2}^{4} kydydx$$



Integrating, we obtain:

$$k\int_{-2}^{2}\frac{1}{2}y^2\Big|_{x^2}^{4} dx = \frac{k}{2}\int_{-2}^{2} 16 - x^4 dx =$$

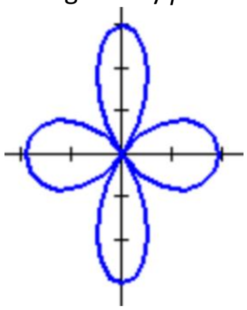
$$\frac{k}{2}\left[16x - \frac{1}{5}x^5\right]_{-2}^{2} =$$

$$\frac{k}{2}\left[32 - \frac{32}{5} - (-32) + \left(-\frac{32}{5}\right)\right] = \frac{k}{2}\left[64 - \frac{64}{5}\right] = \frac{128k}{5}$$

ii. Polar Coordinates

We may also wish to find total mass in polar coordinates.

Example 3. We wish to find the total mass of one petal of $r = 3\cos(2\theta)$, with density per unit area given by $\rho = kr$.



Since the density depends on only the radius, it does not matter which petal we select. To obtain bounds of integration, we find where $3\cos(2\theta) = 0$. We find that for the first petal $\theta = \frac{\pi}{4}, -\frac{\pi}{4}$. The limits of the radial direction are the rose and the origin. Thus we obtain the integral

$$M = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{3\cos(2\theta)} kr \cdot r dr d\theta$$

Integrating, we obtain

$$k\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{3}r^3\Big|_0^{3\cos(2\theta)} d\theta = \frac{k}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 27\cos^3(2\theta)\, d\theta \ = 9k\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta\,(1 - \sin^2 2\theta) d\theta$$

$$= 9k\left[\frac{1}{2}\sin 2\theta - \frac{1}{6}\sin^3 2\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 9k\left[\frac{1}{2}(1) - \frac{1}{6}(1)^3 - \frac{1}{2}(-1) + \frac{1}{6}(-1)^3\right] = 9k\left[1 - \frac{1}{3}\right] = 6k$$

Practice Problems.

Find the mass of the lamina with the given boundaries and density function.

c. $\rho(x, y) = kxy, y = x^3, y = \sqrt{x}, x > 0$
d. $\rho(r, \theta) = k\theta, r = 4\sin\theta$
e. $\rho(x, y) = kx^2, y = \sqrt{1 - x^2}, y = \sqrt{4 - x^2}, y > 0$

3. Mass of a lamina over a surface

To find the mass of a lamina over a surface, we set up a surface integral of the form $\iint \rho(x, y) dS$.

Example 4. Find the mass of the lamina over the surface $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 16$, bounded by the coordinate planes, whose density varies according to the function $\rho(x, y) = x^2 y$.

The coordinate planes restrict is to the first quadrant. First we will find the normal vector and its magnitude for dS. The surface is $z = x^2 + y$^2. Putting everything on one side of the equation we get $G(x, y, z) = x^2 + y^2 - z$, and $\nabla G = \langle 2x, 2x, -1\rangle$. The magnitude is $\|\vec{N}\| = \|\nabla G\| = \sqrt{4x^2 + 4x^2 + 1}$, so $dS = \sqrt{4x^2 + 4y^2 + 1}dA$. Because of the surface and the dS term, we will convert to polar coordinates to integrate.

$$\int_0^{\pi/2}\int_0^4 (r^2\cos^2\theta\, r\sin\theta)\sqrt{4r^2 + 1} r dr d\theta = \int_0^{\pi/2}\int_0^4 (r^4\cos^2\theta\sin\theta)\sqrt{4r^2 + 1} dr d\theta$$

We can switch the order of integration to integrate θ first, and then use trig substitution or numerical integration to complete the mass calculation.

$$\int_0^4 -\frac{1}{3}\cos^3\theta\Big|_0^{\frac{\pi}{2}} r^4\sqrt{4r^2+1}dr = \frac{1}{3}\int_0^4 r^4\sqrt{4r^2+1}dr = \frac{3\ln(\sqrt{65}+8)+263144\sqrt{65}}{4608} \approx 460.404$$

Practice Problems.
Find the mass of the lamina over the indicated surface.

f. $S: z = 4 - x^2 - y^2, z \geq 0, \rho(x,y) = x^2y^2$
g. $S: z = x^2 - y^2, inside\ x^2 + y^2 = 9, y \geq 0, \rho(x,y) = y$
h. $S: z = 3x + 4y - 12, first\ octant, \rho(x,y) = x + y$

4. Mass of a volume.

i. Rectangular Coordinates

To find the mass of a volume in rectangular coordinates, we will need to integrate the density function over the volume.

Example 5. Find the mass of the volume bounded by the plane $z = 12 - 4x - 2y$ and the coordinate planes, with density $\rho(x,y,z) = xyz$.

Integrate $\int_0^3 \int_0^{6-2x} \int_0^{12-4x-2y} xyz\,dzdydx$, where the boundaries of the integral are the boundaries of the region (the plane $z = 0\ and\ z = 12 - 4x - 2y$ are the limits for z, then set $z = 0$, and solve for y to find the limits of y to be $y = 0\ and\ y = 6 - 2x$, and finally, set both $z = 0, y = 0$ to find the limit for x).

Integrating this we get:

$$\int_0^3 \int_0^{6-2x} \frac{1}{2}xyz^2|_0^{12-4x-2y} dydx$$
$$= \frac{1}{2}\int_0^3 \int_0^{6-2x} xy(16x^2 + 16xy - 96x + 4y^2 - 48y + 144)dydx$$
$$= \int_0^3 \int_0^{6-2x} 8x^3y + 8x^2y^2 - 48x^2y + 2xy^3 - 24xy^2 + 72xydydx$$
$$= \int_0^3 2x^3y^2 + \frac{8}{3}x^2y^3 - 24x^2y^2 + \frac{1}{2}xy^4 - 8xy^3 + 36xy^2\Big|_0^{6-2x} dx$$
$$= \int_0^3 -\frac{8}{3}x^5 + 32x^4 + 144x^3 - 288x^2 + 216xdx$$
$$= -\frac{4}{9}x^6 + \frac{32}{5}x^5 + 36x^4 - 96x^3 + 108x^2\Big|_0^3 = \frac{324}{5}$$

ii. Cylindrical Coordinates

Your region may begin in cylindrical coordinates, or it may be converted there for easier integrating.

Example 6. Find the mass of the volume bound by the xy-plane and the surface $z = 16 - x^2 - y^2$, with the density given by $\rho(x,y,z) = \sqrt{x^2+y^2}$.

Since the region of intersection with the plane $z = 0$ is a circle, we will convert to cylindrical coordinates, with $z = 16 - r^2$ and $\rho = r$.

$$\int_0^{2\pi}\int_0^4\int_0^{16-r^2} r \cdot r dz dr d\theta = \int_0^{2\pi}\int_0^4\int_0^{16-r^2} r^2 dz dr d\theta =$$
$$\int_0^{2\pi}\int_0^4 r^2 z|_0^{16-r^2} dr d\theta = \int_0^{2\pi}\int_0^4 16r^2 - r^4 dr d\theta =$$
$$\int_0^{2\pi} \frac{16}{3}r^3 - \frac{1}{5}r^5\Big|_0^4 d\theta = \int_0^{2\pi}\frac{2048}{15}d\theta = \frac{4096}{15}\pi$$

iii. **Spherical Coordinates**

The thing to be careful about in spherical is that ρ is used for both density (function), and the distance from the origin (variable). Try not to confuse the two.

Example 7. Find the total mass of the sphere $\rho = 3$, in the first octant, with density $\rho(\rho, \phi, \theta) = \rho\cos\phi$.

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^3 \rho\cos\phi \cdot \rho^2 \sin\phi \, d\rho d\theta d\phi = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^3 \rho^3\cos\phi\sin\phi \, d\rho d\theta d\phi$$
$$= \int_0^{\pi/2}\int_0^{\pi/2} \frac{1}{4}\rho^4\cos\phi\sin\phi\Big|_0^3 d\theta d\phi = \int_0^{\pi/2}\int_0^{\pi/2}\frac{81}{4}\cos\phi\sin\phi \, d\theta d\phi$$
$$= \int_0^{\pi/2}\frac{81\pi}{2}\cos\phi\sin\phi \, d\phi = \frac{81\pi}{4}\sin^2\phi|_0^{\pi/2} = \frac{81\pi}{4}$$

Practice Problems.

Find the mass over the indicated region with the given density.

i. The rectangular solid with a vertex at the origin and another at (3,4,7), with density $\rho(x, y, z) = x^2z$.
j. The cone $z = \sqrt{x^2 + y^2}$ bounded by the hemisphere $z = \sqrt{25 - x^2 - y^2}$, with density $\rho(x, y, z) = y^2$.
k. The tetrahedron bounded by the coordinate planes and $3x + 2y + 6z = 6$ with density $\rho(x, y, z) = yz$.
l. The cylinder bounded in the xy-plane by one petal of $r = 4\cos(2\theta)$, and by $z = 0, and\ z = 4 - 2x$, with density $\rho(r, \theta, z) = r$.

5. **Center of mass.**

The calculate the center of mass, we need to set up two-three additional equations (one per dimension) to find the coordinates of the center. These equations are called the moments, and in each case, we will need to divide by the total mass we calculated before to obtain the coordinates.

The equations below exist in rectangular coordinates, and output coordinates in rectangular coordinates. If we need to integrate in spherical or cylindrical, we do not have the corresponding formulas in cylindrical or spherical. We can convert the equations we have to obtain values in rectangular, and if needed, then convert the coordinates we obtain to cylindrical or spherical.

i. Center of Mass in the Plane.

When calculating the center of mass of the plane, we have two equations (note that the notation can be a little confusing).

Moment of mass around the x-axis: $M_x = \int_a^b \int_{f(x)}^{g(x)} y\rho(x,y)\,dydx$

Moment of mass around the y-axis: $M_y = \int_a^b \int_{f(x)}^{g(x)} x\rho(x,y)\,dydx$

The center of mass is given by $\left(\frac{M_y}{M}, \frac{M_x}{M}\right)$, where M is the total mass. Note that the subscripts may be opposite what you would expect. The moment from the x-axis is the y-coordinate direction, and the moment from the y-axis is the x-coordinate direction.

The limits of integrate of our moment integrals are the same as we used for the total mass. Only the function we are integrating has changed.

Example 8. Find the center of mass for the region in Example 2.
In Example 2, we found the total mass to be

$$M = \int_{-2}^{2} \int_{x^2}^{4} kydydx = \frac{128k}{5}$$

The moments of mass are:

$$M_x = \int_{-2}^{2} \int_{x^2}^{4} y \cdot kydydx = \int_{-2}^{2} \int_{x^2}^{4} ky^2dydx = \frac{512k}{7}$$

$$M_y = \int_{-2}^{2} \int_{x^2}^{4} x \cdot kydydx = \int_{-2}^{2} \int_{x^2}^{4} kxydydx = 0$$

The center of mass is therefore $\left(\frac{0}{\frac{128k}{5}}, \frac{\frac{512k}{7}}{\frac{128k}{5}}\right) = \left(0, \frac{20}{7}\right)$. Notice that constant multipliers like k will cancel out, so you can ignore those since they don't affect the outcome.

Note that we obtained zero for one of the coordinates. While zero for a mass would be problematic, zero as a moment is fine when the region spans that coordinate. In this case, the region is symmetric, and the mass is changing according to y alone, so it's to be expected that the geometric center of the region, $x = 0$ would be the center of mass. As for the y-coordinate, the center of mass is above the center of the region (the geometric center is $y = 2$, and this also makes sense since the density is greater for larger values of y, that would naturally pull the center of mass in the direction of greater density.

Example 9. Set up the integrals to compute the center of mass for Example 3.
The total mass we obtained above is:

$$M = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{0}^{3\cos(2\theta)} kr \cdot rdrd\theta = 6k$$

To find the moments of mass, since this function is in polar coordinates, we must convert the x and y we multiply inside the integral into polar also.

$$M_x = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{3\cos(2\theta)} y \cdot kr \cdot r dr d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{3\cos(2\theta)} r\sin\theta \cdot kr \cdot r dr d\theta$$
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{3\cos(2\theta)} kr^3 \sin\theta \, dr d\theta$$
$$M_y = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{3\cos(2\theta)} x \cdot kr \cdot r dr d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{3\cos(2\theta)} r\cos\theta \cdot kr \cdot r dr d\theta$$
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{3\cos(2\theta)} kr^3 \cos\theta \, dr d\theta$$

These integrals produce coordinates in rectangular coordinates. *There are no corresponding integrals in polar like M_r or M_{theta}.*

Practice Problems.
Find the center of mass of the lamina with the given boundaries and density function.

m. $\rho(x,y) = kxy, y = x^3, y = \sqrt{x}, x > 0$
n. $\rho(r,\theta) = k\theta, r = 4\sin\theta$
o. $\rho(x,y) = kx^2, y = \sqrt{1-x^2}, y = \sqrt{4-x^2}, y > 0$

i. **Center of Mass of a Volume.**

Three moments are needed to calculate the center of mass of a volume. As before, they are only in rectangular coordinates, and they mark moments from the coordinate planes.

Moment of mass from the xy-plane: $M_{xy} = \int_a^b \int_{f(x)}^{g(x)} \int_{p(x,y)}^{q(x,y)} z\rho(x,y,z) dz\, dy\, dx$

Moment of mass from the yz-plane: $M_{yz} = \int_a^b \int_{f(x)}^{g(x)} \int_{p(x,y)}^{q(x,y)} x\rho(x,y,z) dz\, dy\, dx$

Moment of mass from the xz-plane: $M_{xz} = \int_a^b \int_{f(x)}^{g(x)} \int_{p(x,y)}^{q(x,y)} y\rho(x,y,z) dz\, dy\, dx$

The center of mass is given by $\left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$, where M is the total mass. I find it helpful to remember which integral goes with which coordinate by looking at the integral itself. The variable the moment belongs to is the same as the variable multiplied by the density. As with the 2D case, the limits of integration match what you did with the total mass.

Example 10. Set up the integrals to find the center of mass from Example 5.

The total mass integral was:

$$M = \int_0^3 \int_0^{6-2x} \int_0^{12-4x-2y} xyz\, dzdydx$$

Our moments of mass are, therefore:

$$M_{xy} = \int_0^3 \int_0^{6-2x} \int_0^{12-4x-2y} z \cdot xyz\, dzdydx = \int_0^3 \int_0^{6-2x} \int_0^{12-4x-2y} xyz^2\, dzdydx$$
$$M_{xz} = \int_0^3 \int_0^{6-2x} \int_0^{12-4x-2y} y \cdot xyz\, dzdydx = \int_0^3 \int_0^{6-2x} \int_0^{12-4x-2y} xy^2z\, dzdydx$$
$$M_{yz} = \int_0^3 \int_0^{6-2x} \int_0^{12-4x-2y} x \cdot xyz\, dzdydx = \int_0^3 \int_0^{6-2x} \int_0^{12-4x-2y} x^2yz\, dzdydx$$

Example 11. Set up the center of mass integrals for the region in Example 6.
Our mass integral was:

$$M = \int_0^{2\pi} \int_0^4 \int_0^{16-r^2} r^2 dzdrd\theta$$

Recall that we are still producing coordinates in rectangular coordinates, but since we integrating in cylindrical, we replace $x = r\cos\theta$, and $y = r\sin\theta$. Only the z-coordinate remains unchanged.

$$M_{xy} = \int_0^{2\pi} \int_0^4 \int_0^{16-r^2} z \cdot r^2 dzdrd\theta$$
$$M_{xz} = \int_0^{2\pi} \int_0^4 \int_0^{16-r^2} y \cdot r^2 dzdrd\theta = \int_0^{2\pi} \int_0^4 \int_0^{16-r^2} r\sin\theta \cdot r^2 dzdrd\theta$$
$$= \int_0^{2\pi} \int_0^4 \int_0^{16-r^2} r^3 \sin\theta\, dzdrd\theta$$
$$M_{yz} = \int_0^{2\pi} \int_0^4 \int_0^{16-r^2} x \cdot r^2 dzdrd\theta = \int_0^{2\pi} \int_0^4 \int_0^{16-r^2} r\cos\theta \cdot r^2 dzdrd\theta$$
$$= \int_0^{2\pi} \int_0^4 \int_0^{16-r^2} r^3 \cos\theta\, dzdrd\theta$$

Example 12. Set up the center of mass integrals for the region in Example 7.
Our mass integral was:

$$M = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 \rho^3 \cos\phi \sin\phi\, d\rho d\theta d\phi$$

As with cylindrical, we use the same moment integrals, replacing $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$.

$$M_{xy} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 z \cdot \rho^3 \cos\phi \sin\phi \, d\rho d\theta d\phi$$
$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 \rho \cos\phi \cdot \rho^3 \cos\phi \sin\phi \, d\rho d\theta d\phi$$
$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 \rho^4 \cos^2\phi \sin\phi \, d\rho d\theta d\phi$$
$$M_{xz} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 y \cdot \rho^3 \cos\phi \sin\phi \, d\rho d\theta d\phi$$
$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 \rho \sin\theta \sin\phi \cdot \rho^3 \cos\phi \sin\phi \, d\rho d\theta d\phi$$
$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 \rho^4 \cos\phi \sin^2\phi \sin\theta \, d\rho d\theta d\phi$$

$$M_{yz} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 x \cdot \rho^3 \cos\phi \sin\phi \, d\rho d\theta d\phi$$
$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 \rho \, cos\theta \sin\phi \cdot \rho^3 \cos\phi \sin\phi \, d\rho d\theta d\phi$$
$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 \rho^4 \cos\phi \sin^2\phi \cos\theta \, d\rho d\theta d\phi$$

Practice Problems.

Find the center of mass for the indicated region with the given density.

p. The rectangular solid with a vertex at the origin and another at (3,4,7), with density $\rho(x, y, z) = x^2 z$.

q. The cone $z = \sqrt{x^2 + y^2}$ bounded by the hemisphere $z = \sqrt{25 - x^2 - y^2}$, with density $\rho(x, y, z) = y^2$.

r. The tetrahedron bounded by the coordinate planes and $3x + 2y + 6z = 6$ with density $\rho(x, y, z) = yz$.

s. The cylinder bounded in the xy-plane by one petal of $r = 4\cos(2\theta)$, and by $z = 0, and\ z = 4 - 2x$, with density $\rho(r, \theta, z) = r$.