MAT 202, Exam #1, Part I, Fall 2016 Name KEY

Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. When you are finished with this portion of exam, get Part II.

1. Determine if each statement is True or False. (1 points each)

a. T F A system of two linear equations in three variables is always consistent.

b. T F Two systems of linear equations are equivalent when they have the same solution set.

c. T F A 5×6 matrix has six columns.

d. T F If the row-echelon form of the augmented matrix of a system of linear equations contains the row $[1 \quad 0 \quad 0 \quad 0 \quad 0]$, then the original system is inconsistent.

e. T F Every matrix has a unique reduced row-echelon form.

f. T F A consistent system of linear equations can have infinitely many solutions.

g. T F The system $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ can be expressed as a linear combination of the columns of A, where the coefficients of the linear combination are a solution to the system.

h. T F If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product AB is an $m \times p$ matrix.

i. T F Matrix multiplication is associative.

j. T F If the matrices A, B, C satisfy $AB = AC$ and A is invertible, then $B = C$.

k. T F The identity matrix is an elementary matrix.

l. T F A square matrix is nonsingular when it can be written as the product of elementary matrices.

m. T F The transpose of the sum of matrices is equal to the sum of the transposes of the matrices.

n. T F If A and B are nonsingular $n \times n$ matrices and A is invertible, then $(ABA^{-1})^2 = AB^2A^{-1}$.

2. Solve the system of equations $\begin{cases} x_1 - 2x_2 = -6 \\ 3x_1 + 2x_2 = 12 \end{cases}$ by writing the system as an augmented matrix and row-reducing by hand. (5 points)

$\left[\begin{array}{cc|c} 1 & -2 & -6 \\ 3 & 2 & 12 \end{array}\right]$ $-3R_1 + R_2 \to R_1$

$\left[\begin{array}{cc|c} 1 & -2 & -6 \\ 0 & 8 & 30 \end{array}\right]$ $\frac{1}{8}R_2 \to R_2$

$\left[\begin{array}{cc|c} 1 & -2 & -6 \\ 0 & 1 & 15/4 \end{array}\right]$

$2R_2 + R_1 \to R_1$

$\left[\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & 15/4 \end{array}\right]$

$\vec{x} = \begin{bmatrix} 3/2 \\ 15/4 \end{bmatrix}$

3. Solve the system $\begin{cases} x_1 - 2x_2 - 8x_3 = 2 \\ 2x_1 + 3x_2 \qquad = 5 \end{cases}$ and write the solution in parametric form. (5 points)

$\left[\begin{array}{ccc|c} 1 & -2 & -8 & 2 \\ 2 & 3 & 0 & 5 \end{array}\right]$ $-2R_1 + R_2 \to R_2$ $\left[\begin{array}{ccc|c} 1 & -2 & -8 & 2 \\ 0 & 7 & 16 & 1 \end{array}\right]$

$\frac{1}{7}R_2 \to R_2$ $\left[\begin{array}{ccc|c} 1 & -2 & -8 & 2 \\ 0 & 1 & 16/7 & 1/7 \end{array}\right]$ $2R_2 + R_1 \to R_1$

$\left[\begin{array}{ccc|c} 1 & 0 & -24/7 & 16/7 \\ 0 & 1 & 16/7 & 1/7 \end{array}\right]$

$x_1 = 24/7\, x_3 + 16/7$
$x_2 = -16/7\, x_3 + 1/7$
$x_3 = \quad x_3$

$\vec{x} = \begin{bmatrix} 24 \\ -16 \\ 7 \end{bmatrix} t + \begin{bmatrix} 16/7 \\ 1/7 \\ 0 \end{bmatrix}$

4. Explain why a homogeneous system is always consistent. Define the term *trivial solution* in your explanation. (3 points)

every homogeneous system always has the trivial solution where all of the variables are equal to zero.
Since there is always at least one solution, the system is always consistent.

5. For the matrices $A = \begin{bmatrix} 5 & 3 \\ 4 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & 2 & 1 \\ -5 & 0 & 2 \\ 4 & -1 & -4 \end{bmatrix}$. Calculate the following matrices. If the operation is not defined, explain why not. (4 points each)

a. $3A - B^T$

$$\begin{bmatrix} 15 & 9 \\ 12 & -3 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 13 & 12 \\ 13 & -7 \end{bmatrix}$$

b. $D^T C$

$$\underset{3\times 3}{\begin{bmatrix} 1 & -5 & 4 \\ 2 & 0 & -1 \\ 1 & 2 & -4 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 1 & 0 \end{bmatrix}} = \begin{bmatrix} 2+5+4 & 1-15+0 \\ 4+0-1 & 2+0+0 \\ 2-2-4 & 1+6+0 \end{bmatrix} = \begin{bmatrix} 11 & -14 \\ 3 & 2 \\ -4 & 7 \end{bmatrix}$$

c. CB

$$\underset{3\times 2}{\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 1 & 0 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}} = \begin{bmatrix} 4-3 & -2+4 \\ -2-9 & 1+12 \\ 2+0 & -1+0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -11 & 13 \\ 2 & -1 \end{bmatrix}$$

d. AC

$$\underset{2\times 2}{\begin{bmatrix} 5 & 3 \\ 4 & -1 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 1 & 0 \end{bmatrix}}$$

not defined

e. B^{-1}

$$\frac{1}{8-3}\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 4/5 & 1/5 \\ 3/5 & 2/5 \end{bmatrix}$$

6. Define an *elementary matrix* and give at least three examples and say what they do. (4 points)

an elementary matrix is formed from one row operation applied to the identity matrix

for instance $R_1 \leftrightarrow R_2$ is given by $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for 3x3 matrices

$R_1 + R_2 \to R_2$ $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\frac{1}{4}R_1 \to R_1 \Rightarrow \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

all elementary matrices are invertible

7. Find the LU factorization of $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$. (4 points)

$-3R_1 + R_2 \to R_2$ $\quad \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = E_1$ $\qquad E_1^{-1} = L$

$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ $\quad U = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

8. What does it mean for a system to be underdetermined? Give an example. (3 points)

it means there are fewer equations than there are variables

$$\begin{cases} x_1 + x_2 + x_3 = 5 \\ x_1 + 2x_2 - 3x_3 = 11 \end{cases}$$

MAT 202, Exam #1, Part II, Fall 2016 Name ______________________________

Instructions: Show all work. You **may** use a calculator on this portion of the exam. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question. Provide explanations where requested.

1. Graphically solve the system of equations $\begin{cases} -5.2x + 1.3y = 1.25 \\ 0.8x - 0.2y = 1.8 \end{cases}$. Write the solution in vector form, and round the solutions to 3 decimal places if needed. Classify the system as consistent or inconsistent; classify the solution as independent or dependent. Sketch the graph. (6 points)

$y = \frac{(5.2x + 1.25)}{1.3}$ $y = \frac{.8x - 1.8}{.2}$

the system is inconsistent it has no solution since the lines are parallel, they do not intersect

2. Find a cubic model for the data in the table below.

Year (after 2000)	**1**	**3**	**7**	**10**
Profit (millions)	61	113	127	102

Assuming this models holds for all years between 2001 and 2010, what was the profit in 2005? (5 points)

$ax^3 + bx^2 + cx + d = y$

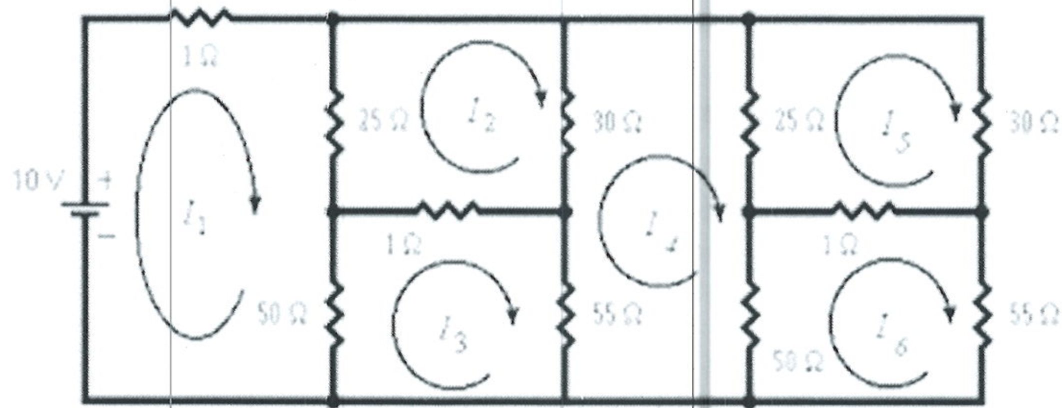
$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 61 \\ 27 & 9 & 3 & 1 & 113 \\ 343 & 49 & 7 & 1 & 127 \\ 1000 & 100 & 10 & 1 & 102 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 173/756 \\ 0 & 1 & 0 & 0 & -2369/378 \\ 0 & 0 & 1 & 0 & 36359/756 \\ 0 & 0 & 0 & 1 & 341/18 \end{array}\right] \begin{array}{l} \approx .2288 \\ \approx -6.2672 \\ \approx 48.094 \\ \approx 18.94 \end{array}$$

$y(x) = \frac{173}{756}x^3 - \frac{2369}{378}x^2 + \frac{36,359}{756}x + \frac{341}{18}$

$y(5) = 131.33$ million dolla

this is reasonable

3. Set up and solve the loop circuit diagram below. Round your values for the currents to three significant digits. (7 points)



$$76I_1 - 25I_2 - 50I_3 = 10$$
$$-25I_1 + 56I_2 - I_3 - 30I_4 = 0$$
$$-50I_1 - I_2 + 106I_3 - 55I_4 = 0$$
$$-30I_2 - 55I_3 + 160I_4 - 25I_5 - 50I_6 = 0$$
$$-25I_4 + 56I_5 - I_6 = 0$$
$$-50I_4 - I_5 + 106I_6 = 0$$

$$\left[\begin{array}{cccccc|c} 76 & -25 & -50 & 0 & 0 & 0 & 10 \\ -25 & 56 & -1 & -30 & 0 & 0 & 0 \\ -50 & -1 & 106 & -55 & 0 & 0 & 0 \\ 0 & -30 & -55 & 160 & -25 & -50 & 0 \\ 0 & 0 & 0 & -25 & 56 & -1 & 0 \\ 0 & 0 & 0 & -50 & -1 & 106 & 0 \end{array}\right]$$

$$\vec{I} = \begin{bmatrix} .478 \\ .348 \\ .353 \\ .239 \\ .109 \\ .114 \end{bmatrix}$$

4. For $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 2 \\ -1 & 0 & 4 \end{bmatrix}$ and $f(x) = x^2 - 4x + 7$, find $f(A)$. (4 points)

$$A^2 - 4A + 7I = \begin{bmatrix} 7 & 0 & 0 \\ -2 & 6 & 1 \\ -2 & -1 & 8 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 8 & 4 & -4 \\ 10 & 7 & 9 \\ -6 & -1 & 17 \end{bmatrix}$$

$$-4A = \begin{bmatrix} -8 & -4 & 4 \\ -12 & -8 & -8 \\ 4 & 0 & -16 \end{bmatrix}$$

$$7I = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

5. Solve the system $\begin{cases} x_1 - x_2 + 2x_3 + x_4 = -2 \\ 2x_1 + x_2 - 2x_3 - 3x_4 = 6 \\ x_1 - 3x_2 + x_3 + 2x_4 = -3 \\ x_1 + 2x_2 - x_3 + 3x_4 = -3 \end{cases}$ by inverse methods. You may find the inverse in your calculator, but multiply $A^{-1}\vec{b}$ by hand. (5 points)

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -2 & -3 \\ 1 & -3 & 1 & 2 \\ 1 & 2 & -1 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2/7 & 11/42 & 1/14 & 5/42 \\ 3/14 & -1/84 & -9/28 & 11/84 \\ 1/2 & -1/12 & -1/4 & -1/12 \\ -1/14 & -3/28 & 3/28 & 5/28 \end{bmatrix}$$

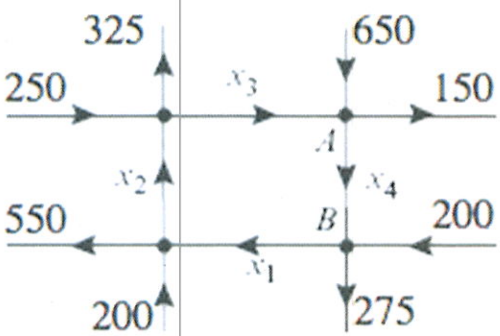
$$\frac{1}{84}\begin{bmatrix} 24 & 22 & 6 & 10 \\ 18 & -1 & -27 & 11 \\ 42 & -7 & -21 & -7 \\ -6 & -9 & 9 & 15 \end{bmatrix}\begin{bmatrix} -2 \\ 6 \\ -3 \\ -3 \end{bmatrix} = \frac{1}{84}\begin{bmatrix} 36 \\ 6 \\ -42 \\ -114 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 1/14 \\ -1/2 \\ -19/14 \end{bmatrix}$$

6. Encode the message WHAT WE THINK WE BECOME with the matrix $A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & -1 & -1 \\ 5 & 2 & 1 \end{bmatrix}$ using 0 as a space and the corresponding position (number) in the alphabet for letters. (5 points)

$$\begin{bmatrix} 23 & 8 & 1 \\ 20 & 0 & 23 \\ 5 & 0 & 20 \\ 8 & 9 & 14 \\ 11 & 0 & 23 \\ 5 & 0 & 2 \\ 5 & 3 & 15 \\ 13 & 5 & 0 \end{bmatrix}$$

73 40 16 195 86 43 120 50
25 75 35 13 159 68 34 36
14 7 86 37 17 37 21 8

7. Set up a matrix for the traffic flow graph below. Solve the system. If the system is dependent, choose a value for the free variable that makes all values positive. What does it mean if the values are negative? (5 points)



$250 + x_2 = x_3 + 325$

$x_3 + 650 = x_4 + 150$

$x_2 + 550 = x_1 + 200$

$x_1 + 275 = x_4 + 200$

$x_2 - x_3 = 75$

$x_3 - x_4 = -500$

$-x_1 + x_2 = -350$

$x_1 - x_4 = -75$

$$\left[\begin{array}{cccc|c} 0 & 1 & -1 & 0 & 75 \\ 0 & 0 & 1 & -1 & -500 \\ -1 & 1 & 0 & 0 & -350 \\ 1 & 0 & 0 & -1 & -75 \end{array}\right] \Rightarrow \text{rref} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -75 \\ 0 & 1 & 0 & -1 & -425 \\ 0 & 0 & 1 & -1 & -500 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = x_4 - 75$

$x_2 = x_4 - 425$

$x_3 = x_4 - 500$

$x_4 = x_4$

if we want to keep the arrows pointing in the directions shown on the graph, x_4 must be ≥ 500 otherwise, negative values mean traffic flow is reversed

8. A medical researcher is studying the spread of a virus in a population of 1000 laboratory mice. During any week, there is an 80% chance that an infected mouse will overcome the virus, and during the same week there is a 10% chance that a non-infected mouse will become infected. One hundred mice are currently infected with the virus. How many will be infected in the next week? The next two weeks? Set up the stochastic matrix and the initial state vector to calculate the result. (7 points)

$$\begin{array}{c|cc} & I & H \\ \hline I & .2 & .1 \\ H & .8 & .9 \end{array}$$

$\vec{X}_0 = \begin{bmatrix} 100 \\ 900 \end{bmatrix}$ next week $\vec{X}_1 = \begin{bmatrix} 110 \\ 890 \end{bmatrix}$ $\vec{X}_2 = \begin{bmatrix} 111 \\ 889 \end{bmatrix}$

week 2

9. An industrial system has two industries with the following input requirements: a) to produce \$1.00 of industry A's output, industry A requires \$0.30 of its own output and \$0.40 of industry B's product; b) to produce \$1.00 of industry B's product, industry B requires \$0.20 of its own product and \$0.40 of industry A's product. Find the input-output model for this system. Then solve for X in the equation $X = DX + E$ if $E = \begin{bmatrix} 10{,}000 \\ 20{,}000 \end{bmatrix}$. (5 points)

$D = \begin{bmatrix} .3 & .4 \\ .4 & .2 \end{bmatrix}$ $\quad (I-D)X = E$

$I - D = \begin{bmatrix} .7 & -.4 \\ -.4 & .8 \end{bmatrix}$

$\left[\begin{array}{cc|c} .7 & -.4 & 10{,}000 \\ -.4 & .8 & 20{,}000 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 40{,}000 \\ 0 & 1 & 45{,}000 \end{array}\right]$

$A = 40{,}000$

$B = 45{,}000$

10. Find two matrices that are idempotent, i.e. two matrices such that $A^2 = A$. (3 points)

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

answers will vary