MAT 202, Exam #2, Part I, Fall 2016

Name KEY

Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. When you are finished with this portion of exam, get Part II.

1. Determine if each statement is True or False. (1 point each)

a. T F The ij-cofactor of a square matrix A is defined by deleting the ith column and jth row of the matrix. ith row, jth column

b. T F When expanding by cofactors, you need not evaluate cofactors of zero entries.

c. T F Multiplying a column of a matrix by a nonzero constant results in changing the determinant by the same nonzero constant.

d. T F If two matrices are column-equivalent, then their determinants are the same. Some column operations change determinant

e. T F If A is invertible, then the determinant of A^{-1} is the reciprocal of the determinant of A.

f. T F If A and B are square matrices of order n, and $|A| = |B|$, then $|AB| = |A^2|$.

g. T F If A is a square matrix of order n, then $\det(A) = -\det(A^T)$. not negative

h. T F In Cramer's Rule, the denominator is the determinant of the matrix formed by replacing the column corresponding to the variable being solved for with the column representing the constants. numerator

i. T F Two vectors in R^n are equal if and only if their corresponding components are equal.

j. T F A vector space consists of four entities: a set of vectors, a set of scalars, and two operations.

k. T F The set of all first degree polynomials with the standard operations is a vector space. does not have zero vector

l. T F Every vector space V contains at least one subspace that is the zero vector.

m. T F If U, V and W are vector spaces such that W is a subspace of V and U is a subspace of V, then $W = U$.

n. T F If a subset S spans a vector space V, then every vector in V can be written as a linear combination of the vectors in S.

o. T F The dimension of $M_{2,3}$ is six.

p. T F The set of vectors $\left\{\begin{bmatrix}4\\5\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$ is linearly dependent.

independent

q. T F The dimension of a vector space is equal to the number of vectors in any basis for the space.

r. T F The number of pivots plus the number of free variables in a matrix equals the number of rows of the matrix.

Columns

s. T F The column space of a matrix A is equal to the row space of A^T.

t. T F For an 4×1 matrix X, the coordinate matrix $[X]_S$ relative to the standard basis for $M_{4,1}$ is equal to X itself.

2. Find the determinant of the matrix $\begin{bmatrix}2 & -1 & 3\\1 & 4 & 4\\1 & 0 & 2\end{bmatrix}$ by the cofactor method. (5 points)

$$-(-1)\begin{vmatrix}1 & 4\\1 & 2\end{vmatrix} + 4\begin{vmatrix}2 & 3\\1 & 2\end{vmatrix} + 0$$

$$= 1(2-4) + 4(4-3) = -2 + 4 = \boxed{2}$$

3. Use row-reducing methods to find the determinant of $\begin{bmatrix} 1 & -2 & -4 & 7 \\ 3 & 4 & 4 & 5 \\ 3 & 3 & 1 & -1 \\ 4 & 5 & 3 & 2 \end{bmatrix}$. (5 points)

$-3R_1+R_2 \to R_2$
$-3R_1+R_3 \to R_3$
$-4R_1+R_4 \to R_4$

$\begin{bmatrix} 1 & -2 & -4 & 7 \\ 0 & 10 & 16 & -16 \\ 0 & 9 & 13 & -22 \\ 0 & 13 & 19 & -26 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & 16 & -16 \\ 9 & 13 & -22 \\ 13 & 19 & -26 \end{bmatrix}$ $C_2 + C_3 \to C_3$

$\begin{bmatrix} 10 & 16 & 0 \\ 9 & 13 & -8 \\ 13 & 19 & -7 \end{bmatrix}$ $C_3 + C_1 \to C_1$, $C_3 + C_2 \to C_2$ $\begin{bmatrix} 10 & 16 & 0 \\ 1 & 13 & -8 \\ 6 & 19 & -7 \end{bmatrix} \to \begin{bmatrix} 10 & 16 & 0 \\ 1 & 5 & -8 \\ 6 & 12 & -7 \end{bmatrix}$ $-R_2 + R_3 \to R_3$

$\begin{bmatrix} 10 & 16 & 0 \\ 1 & 5 & -8 \\ 5 & 7 & 1 \end{bmatrix}$ $8R_3 + R_2 \to R_2$ $\begin{bmatrix} 10 & 16 & 0 \\ 41 & 61 & 0 \\ 5 & 7 & 1 \end{bmatrix} \Rightarrow \begin{vmatrix} 10 & 16 \\ 41 & 61 \end{vmatrix} = -46$

$610 - 656 =$

4. Use properties of determinants and $A = \begin{bmatrix} 1 & 2 \\ 3 & 11 \end{bmatrix}$, $\det(B) = -4$ to find: (2 points each)

$\det A = 11 - 6 = 5$

a. $\det(AB)$

$5(-4) = -20$

b. $\det(B^3)$

$(-4)^3 = -64$

c. $\det(A^{-1})$

$\frac{1}{5}$

d. $\det(-B)$

$(-1)^2(-4) = -4$

e. $\det(A^T(B^T)^{-1})$

$5\left(\frac{1}{-4}\right) = -\frac{5}{4}$

5. For the vectors $\vec{u} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$, find $2\vec{u} - \vec{w} + 4\vec{v}$. (2 points)

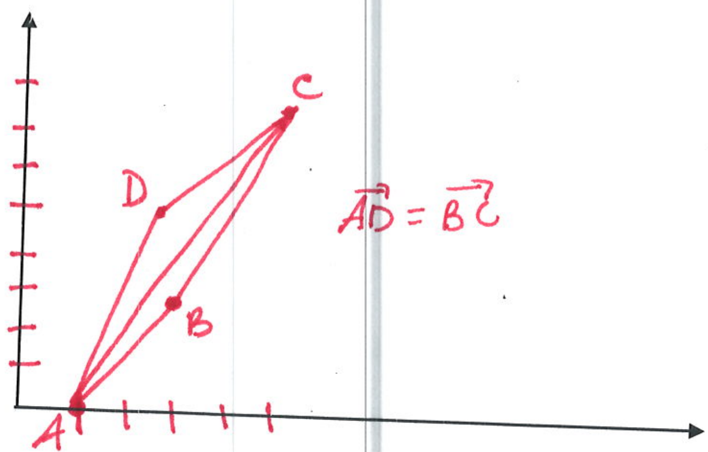
$\begin{bmatrix} 10 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} + \begin{bmatrix} 12 \\ 8 \\ -4 \end{bmatrix} = \begin{bmatrix} 20 \\ 15 \\ -10 \end{bmatrix}$

6. Graph the parallelogram with vertices $A(1,0), B(3,3), C(5,8), D(3,5)$. Find the vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ and explain how this illustrates the parallelogram rule: $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$. (3 points)

$\overrightarrow{AB} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\overrightarrow{AC} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$\overrightarrow{AD} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$



$\overrightarrow{AD} = \overrightarrow{BC}$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

explanations will vary

7. Determine if each of the following sets is a subspace or vector space. If it is a subspace, prove it. If it is not, provide an example of where it fails a property and state which property it fails. (4 points each)

a. $H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b = c^2 \right\}$

it is not

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}$ $(a+d)+(b+e) \stackrel{?}{=} (c+f)^2$ but $(c+f)^2 \neq c^2+f^2$

$d+e=f^2$ $(a+b)+(d+e) \stackrel{?}{=} c^2+f^2$

b. The set of all functions whose x-intercept is the origin (i.e. $f(0) = 0$).

it is a subspace $f(x)=0$ has $f(0)=0$ in set

$f(0)+g(0)=(f+g)(0)=0$ in set

$kf(0)=k(0)=0$ in set ✓

c. $W = \left\{ \begin{bmatrix} a & b \\ c & 1-b \end{bmatrix}, a.b, c\ real \right\}$

not a subspace

$a,b,c=0$ $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ no zero vector

if $b=1$ $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

fails all three tests so answers will vary

8. Determine by inspection, if each set of vectors is linearly independent. Explain your reasoning. (2 points each)

a. $\left\{\begin{bmatrix}-2\\2\\3\\5\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-4\\2\\1\end{bmatrix}\right\}$ dependent, zero vector in set

b. $\left\{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}$ independent, pivot in every column

c. $\{2x^2 - x, 2x - 1\}$ independent, 2 vectors, not constant multiples

d. $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\3\\6\end{bmatrix}, \begin{bmatrix}3\\4\\7\end{bmatrix}\right\}$ dependent, first 2 vectors add to third vector

9. Consider the basis for R^3, $B = \left\{\begin{bmatrix}1\\3\\-1\end{bmatrix}, \begin{bmatrix}2\\4\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\-2\end{bmatrix}\right\}$ and $[\vec{x}]_B = \begin{bmatrix}9\\-3\\6\end{bmatrix}$. Write the vector in the standard basis. (3 points)

$P_B[\vec{x}]_B = \vec{x}$

$$\begin{bmatrix}1 & 2 & -1\\3 & 4 & 0\\-1 & 1 & -2\end{bmatrix}\begin{bmatrix}9\\-3\\6\end{bmatrix} = \begin{bmatrix}9-6-6\\27-12+0\\-9-3-12\end{bmatrix} = \begin{bmatrix}-3\\15\\-24\end{bmatrix}$$

MAT 202, Exam #2, Part II, Fall 2016 Name ____________________

Instructions: Show all work. You **may** use a calculator on this portion of the exam. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question. Provide explanations where requested.

1. Use Cramer's Rule to solve the system $\begin{cases} 2x_1 + 3x_2 + 5x_3 = 4 \\ 3x_1 + 5x_2 + 9x_3 = 7 \\ 5x_1 + 9x_2 + 17x_3 = 13 \end{cases}$. Show all the required matrices. You may use your calculator to find the required determinants. State your final solution as a vector. (5 points)

$A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{bmatrix} \quad \det A = 0$

this system cannot be solved by Cramer's rule

the solution is either dependent or inconsistent

$\text{rref} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = 2x_3 - 1$
$x_2 = -3x_3 + 2$
$x_3 = x_3$

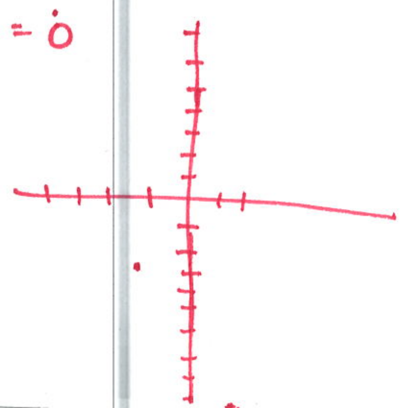
$\vec{x} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

2. Find the area of the triangle given by the vertices $(-1,-3), (-4,7), (2,-13)$. (3 points)

$\frac{1}{2}\left|\det\left(\begin{bmatrix} -1 & -3 & 1 \\ -4 & 7 & 1 \\ 2 & -13 & 1 \end{bmatrix}\right)\right| = \frac{1}{2}(0) = 0$

these lines are collinear

they do not form a triangle



3. Find the volume of the tetrahedron whose corner is defined by the vertices $(3,-1,1)$, $(4,-4,4)$, $(1,1,1)$, $(0,0,1)$. Sketch a graph of the shape. (4 points)

$$\frac{1}{6}\left|\det\left(\begin{bmatrix} 3 & -1 & 1 & 1 \\ 4 & -4 & 4 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}\right)\right| = \frac{1}{6}\left|-12\right| = 2$$

4. Write $\vec{v} = \begin{bmatrix} 2 \\ 5 \\ -4 \\ 0 \end{bmatrix}$ as a linear combination of $\vec{u}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 2 \\ -2 \\ -5 \\ 4 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 6 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 0 \\ -1 \\ -8 \\ -2 \end{bmatrix}$ in two different ways. Explain why this shows that the set of vector $\{\vec{u}_i\}$ is dependent. (3 points)

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 0 & 2 \\ 3 & -2 & -1 & -1 & 5 \\ 2 & -5 & 3 & -8 & -4 \\ 1 & 4 & 6 & -2 & 0 \end{array}\right] \text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$2\vec{u}_1 + \vec{u}_2 - \vec{u}_3$

$2\vec{u}_1 + \vec{u}_4$

answers may vary

5. Find a spanning set for the space spanned by the vectors $\left\{\begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}\right\}$.

(4 points)

$$\text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 118/161 \\ 0 & 1 & 1 & 0 & 0 & -45/161 \\ 0 & 0 & 0 & 1 & 1 & -53/161 \\ 0 & 0 & 0 & 0 & 0 & 74/161 \end{bmatrix}$$

pivots

$$\left\{\begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 2 \\ 4 \end{bmatrix}\right\}$$

6. Determine whether $S = \{t^3 - 1, 2t^2, t + 3, 5 + 2t + 2t^2 + t^3\}$ is a basis for P_3. (3 points)

$$\begin{bmatrix} -1 & 0 & 3 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{rref} \Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

it is not a basis

Since there are only 3 pivots

not independent

does not span

7. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 5 & -1 & -3 & 4 \\ 2 & 3 & 1 & 0 & 7 & 4 \\ 0 & -1 & 11 & 4 & -1 & 12 \\ 8 & 4 & -2 & 6 & 1 & -3 \end{bmatrix}$. Find a *basis* for:

$$\text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -1339/166 & -649/166 \\ 0 & 1 & 0 & 0 & 1347/166 & 623/166 \\ 0 & 0 & 1 & 0 & -201/166 & 93/166 \\ 0 & 0 & 0 & 1 & 424/83 & 199/83 \end{bmatrix}$$

$$x_1 = \frac{1339}{166} x_5 + \frac{649}{166} x_6$$
$$x_2 = -\frac{1347}{166} x_5 - \frac{623}{166} x_6$$
$$x_3 = \frac{201}{166} x_5 - \frac{93}{166} x_6$$
$$x_4 = \frac{-848}{166} x_5 - \frac{398}{166} x_6$$
$$x_5 = x_5$$
$$x_6 = x_6$$

a. The nullspace of A (3 points)

$$\left\{ \begin{bmatrix} 1339 \\ -1347 \\ 201 \\ -848 \\ 166 \\ 0 \end{bmatrix}, \begin{bmatrix} 649 \\ -623 \\ -93 \\ -398 \\ 0 \\ 166 \end{bmatrix} \right\}$$

b. The column space of A (3 points)

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 11 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 6 \end{bmatrix} \right\}$$

c. The row space of A (3 points)

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \\ -1 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \\ 7 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 11 \\ 4 \\ -1 \\ 12 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \\ -2 \\ 6 \\ 1 \\ -3 \end{bmatrix} \right\}$$

d. The rank of A (2 points)

4

8. Find the change of basis matrix to transition from C to B for $B = \left\{ \begin{bmatrix} 9 \\ -3 \\ 15 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 2 \\ -3 \end{bmatrix} \right\}$, $C = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Use it to convert $[\vec{x}]_B = \begin{bmatrix} 1 \\ -3 \\ 4 \\ -2 \end{bmatrix}$ to $[\vec{x}]_C$. (4 points)

$$P_B [\vec{x}]_B = P_C [\vec{x}]_C \qquad P_B = \begin{bmatrix} 9 & 3 & 0 & 3 \\ -3 & 0 & -5 & -4 \\ 15 & 0 & 6 & 2 \\ 4 & 1 & 8 & -3 \end{bmatrix} \qquad P_C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ -1 & 1 & 1 & 1 \end{bmatrix}$$

$$P_C^{-1} P_B [\vec{x}]_B = [\vec{x}]_C$$

$$P_C^{-1} P_B = P_{C \leftarrow B} = \begin{bmatrix} 9 & -3 & 0 & 3 \\ 44/3 & 17/3 & 20/3 & 5 \\ -15/3 & -13/3 & -1/3 & -3 \\ 0/3 & -8/3 & 5/3 & -2 \end{bmatrix}$$

$$[\vec{x}]_C = \begin{bmatrix} -6 \\ 43/3 \\ -40/3 \\ 16/3 \end{bmatrix}$$

9. List 5 different vector spaces (or subspaces) which are 5-dimensional. (5 points)

$\mathbb{R}^5$ $\qquad \begin{bmatrix} a & a \\ b & c \\ d & e \end{bmatrix}$

$\mathbb{P}_4$

answers will vary

$$V = \{ e^{2x}, e^{3x}, e^{4x}, e^{5x}, e^{6x} \}$$

10. Find a basis for the set of each vector space. (6 points)

a. $M_{2,2}$

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

b. P_6

$$\{ 1, x, x^2, x^3, x^4, x^5, x^6 \}$$