MAT 202, Final Exam, Part I, Fall 2016 Name KEY

Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. When you are finished with this portion of exam, get Part II.

1. Determine if each statement is True or False. (2 points each)

a. T F The scalar λ is an eigenvalue of an $n \times n$ matrix A when there exists a vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$.

b. T F If $\vec{u} \cdot \vec{v} > 0$, then the angle θ between $\vec{u}$ and $\vec{v}$ is acute.

c. T F A nilpotent matrix has at least one zero eigenvalue.

d. T F The fact that $n \times n$ matrix A has n distinct eigenvalues does not guarantee that A is diagonalizable. it does guarantee it

e. T F The eigenvectors corresponding to distinct eigenvalues are orthogonal for symmetric matrices.

f. T F If A is an $n \times n$ symmetric matrix, then A has real eigenvalues.

g. T F If one of the eigenvalues of A is 2, then one of the eigenvalues of $A - 5I$ is -3.

h. T F If the row-echelon form of the augmented matrix of a system of linear equations contains the row $[1 \quad 0 \quad 0 \quad 0 \quad 0]$, then the original system is inconsistent. $[0 \ 0 \ 0 \ 0 \ 1]$ would

i. T F A square matrix is nonsingular when it can be written as the product of elementary matrices.

j. T F If A and B are nonsingular $n \times n$ matrices and A is invertible, then $(ABA^{-1})^2 = AB^2A^{-1}$.

k. T F The ij-cofactor of a square matrix A is defined by deleting the ith column and jth row of the matrix. ith row, jth column

l. T F If A is a square matrix of order n, then $\det(A) = -\det(A^T)$. $\det A = \det A^T$

m. T F In Cramer's Rule, the denominator is the determinant of the matrix formed by replacing the column corresponding to the variable being solved for with the column representing the constants. The numerator

n. T F The dimension of $M_{3,3}$ is six. it's 9

o. T F The set of vectors $\left\{ \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$ is linearly independent.

p. T F The dimension of a vector space is equal to the number of vectors in any basis for the space.

q. T F The dot product of two vectors $\vec{u}$ and $\vec{v}$ is another vector represented by $\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 v_1 \\ u_2 v_2 \\ \vdots \\ u_n v_n \end{bmatrix}$.

it is the sum of these elements
it is not a vector

r. T F An orthonormal basis derived by the Gram-Schmidt orthonormalization process does depend on the order of the vectors in the basis.

s. T F For polynomials, the differential operator D_x is a linear transformation from $P_n \rightarrow P_{n-1}$.

t. T F The matrix of a linear transformation is defined by the effects of the transformation on the basis vectors of the space.

2. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$. (8 points)

$(7-\lambda)(4-\lambda)-4=0$

$\lambda^2-11\lambda+28-4=0$

$\lambda^2-11\lambda+24=0$

$(\lambda-3)(\lambda-8)=0$

$\lambda=3,\ \lambda=8$

This matrix is symmetric $ these eigenvalues are real and the eigenvectors are orthogonal.

$\lambda_1=3$ $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ $2x_1+x_2=0$, $x_1=-\frac{1}{2}x_2$, $x_2=x_2$ $\quad \vec{v}_1=\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\lambda_2=8$ $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$ $x_1=2x_2$, $x_2=x_2$ $\quad \vec{v}_2=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

3. For the matrix $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$ with eigenvalues $\lambda_1 = -1, \lambda_2 = 2$ with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. Find the similarity transformation that diagonalizes A and find D. (6 points)

$$P = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

4. Solve the linear system of ODEs given by $\vec{x}' = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix} \vec{x}$. Write the solution in standard form, and plot several sample trajectories. (8 points)

$(-2-\lambda)(1-\lambda) - 4 = 0$

$\lambda^2 + \lambda - 6 = 0$

$(\lambda+3)(\lambda-2) = 0$

$\lambda = -3, \lambda = 2$

$\lambda_1 = -3$

$\begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}$ $\quad x_1 = -4x_2 \quad x_2 = x_2 \quad \vec{v}_1 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

$\lambda_2 = 2$

$\begin{bmatrix} -4 & 4 \\ 1 & -1 \end{bmatrix}$ $\quad x_1 = x_2 \quad x_2 = x_2 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\vec{x} = c_1 \begin{bmatrix} -4 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

5. Find the equilibrium vector for the Markov chain given by the transition matrix $P = \begin{bmatrix} .97 & .22 \\ .03 & .78 \end{bmatrix}$. (7 points)

$P - I = \begin{bmatrix} -.03 & .22 \\ .03 & -.22 \end{bmatrix}$ $x_1 = \frac{.22}{.03} x_2$, $x_2 = x_2$ $\Rightarrow$ $\vec{x} = \begin{bmatrix} 22 \\ 3 \end{bmatrix}$ $22 + 3 = 25$

$\vec{q} = \begin{bmatrix} 22/25 \\ 3/25 \end{bmatrix} = \begin{bmatrix} .88 \\ .12 \end{bmatrix}$

6. Given the vector $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 4 \end{bmatrix}$, find the following: (3 points each)

a. $||\vec{u}||$

$\sqrt{4 + 1 + 16 + 1} = \sqrt{22}$

b. A unit vector in the direction of $\vec{u}$

$\hat{u} = \begin{bmatrix} 2/\sqrt{22} \\ 1/\sqrt{22} \\ 4/\sqrt{22} \\ -1/\sqrt{22} \end{bmatrix}$

c. $\vec{u} \cdot \vec{v}$

$-2 + 2 - 4 - 4 = -8$

d. Are $\vec{u}$ and $\vec{v}$ orthogonal? If not, is the angle between the vectors acute or obtuse?

No, they are not

the angle is obtuse since the dot product is zero.

7. For $W = span\left\{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\3\\-1\end{bmatrix}\right\}$, find the projection of $\vec{y} = \begin{bmatrix}4\\1\\3\\-2\end{bmatrix}$ onto W, and its orthogonal complement in $W^{\perp}$. (8 points)

$$proj_{\vec{w}} y = \left(\frac{4+1+0-2}{1+1+0+1}\right)\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + \left(\frac{0+1+9+2}{0+1+9+1}\right)\begin{bmatrix}0\\1\\3\\-1\end{bmatrix} =$$

$$\left(\cancel{\frac{3}{3}}\right)\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + \frac{12}{11}\begin{bmatrix}0\\1\\3\\-1\end{bmatrix} = \begin{bmatrix}1\\1\\0\\1\end{bmatrix} + \begin{bmatrix}0\\12/11\\36/11\\-12/11\end{bmatrix} = \begin{bmatrix}1\\23/11\\36/11\\-1/11\end{bmatrix} = \vec{y}_{\parallel}$$

$$\vec{y}_{\perp} \text{ in } W^{\perp} = \vec{y} - \vec{y}_{\parallel} = \begin{bmatrix}4\\1\\3\\-2\end{bmatrix} - \begin{bmatrix}1\\23/11\\36/11\\-1/11\end{bmatrix} = \begin{bmatrix}3\\-12/11\\-3/11\\-21/11\end{bmatrix}$$

8. Use any method to find the determinant of $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & 5 & 0 & -1 \\ -1 & 1 & -2 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$. (8 points)

$-R_1+R_2 \to R_2$
$R_1+R_3 \to R_3$
$-R_1+R_4 \to R_4$

$\begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & 3 & 1 & -4 \\ 0 & 3 & -3 & 3 \\ 0 & 0 & 1 & -2 \end{vmatrix}$

$\Rightarrow 1 \begin{vmatrix} 3 & 1 & -4 \\ 3 & -3 & 3 \\ 0 & 1 & -2 \end{vmatrix}$ $-R_1+R_2 \to R_2$ $\begin{vmatrix} 3 & 1 & -4 \\ 0 & -4 & 7 \\ 0 & 1 & -2 \end{vmatrix} \Rightarrow$

$3 \begin{vmatrix} -4 & 7 \\ 1 & -2 \end{vmatrix} = 3(8-7) = \boxed{3}$

9. Use properties of determinants and $\det(A) = 3, B = \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$ to find: (3 points each)

$\det B = -5+4 = -1$

a. $\det(AB)$

$3(-1) = -3$

b. $\det(B^8)$

$(-1)^8 = 1$

c. $\det(-2A)$

$(-2)^2 3 = 4 \cdot 3 = 12$

d. $\det(A^{-1}B^T)$

$\frac{1}{3}(-1) = -\frac{1}{3}$

10. Determine by inspection, if each set of vectors is linearly independent. Explain your reasoning. (3 points each)

a. $\left\{\begin{bmatrix}-2\\2\end{bmatrix}, \begin{bmatrix}1\\4\end{bmatrix}\right\}$ independent ; 2 vectors, not multiples

b. $\left\{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\3\\1\\1\end{bmatrix}, \begin{bmatrix}5\\1\\1\\1\end{bmatrix}\right\}$ independent ; pivot in every column

c. $\{x^2 - 1, 2x^2 - 2, 2x + 5\}$ dependent ; $2x^2-2 = 2(x^2-1)$
2 vectors are multiples

11. For the matrices $A = \begin{bmatrix}-4 & 3\\ 2 & -1\end{bmatrix}, B = \begin{bmatrix}2 & -1\\ -1 & 8\end{bmatrix}, C = \begin{bmatrix}2 & 1\\ -3 & 4\\ 1 & 6\end{bmatrix}, D = \begin{bmatrix}3 & 2 & 1\\ -3 & 0 & 4\\ 4 & -2 & -4\end{bmatrix}$.

Calculate the following matrices. If the operation is not defined, explain why not. (4 points each)

a. DC^T

$$\underset{3\times 3}{\begin{bmatrix}3 & 2 & 1\\ -3 & 0 & 4\\ 4 & -2 & -4\end{bmatrix}} \underset{2\times 3}{\begin{bmatrix}2 & -3 & 1\\ 1 & 4 & 6\end{bmatrix}}$$

not defined
inner dimensions
do not match

b. CA

$$\underset{3\times 2}{\begin{bmatrix}2 & 1\\ -3 & 4\\ 1 & 6\end{bmatrix}} \underset{2\times 2}{\begin{bmatrix}-4 & 3\\ 2 & -1\end{bmatrix}} = \begin{bmatrix}-8+2 & 6-1\\ 12+8 & -9-4\\ -4+12 & 3-6\end{bmatrix} = \begin{bmatrix}-6 & 5\\ 20 & -13\\ 8 & -3\end{bmatrix}$$

MAT 202, Final Exam, Part II, Fall 2016 Name KEY

Instructions: Show all work. You **may** use a calculator on this portion of the exam. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question. Provide explanations where requested.

1. Find e^A for $A = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$. (8 points)

$(-2-\lambda)(1-\lambda)-4 = \lambda^2+\lambda-6=0$

$(\lambda+3)(\lambda-2)=0$

$\lambda=-3, \lambda=2$

$\lambda=-3 \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $\begin{matrix} x_1=-2x_2 \\ x_2=x_2 \end{matrix}$ $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda=2 \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ $\begin{matrix} 2x_1=x_2 \\ x_1=\frac{1}{2}x_2 \\ x_2=x_2 \end{matrix}$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$D=\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$

$P=\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$ $P^{-1}=\frac{1}{-5}\begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}=\frac{1}{5}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$

$e^D=\begin{bmatrix} e^{-3} & 0 \\ 0 & e^2 \end{bmatrix}$

$Pe^DP^{-1}=\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} e^{-3} & 0 \\ 0 & e^2 \end{bmatrix}\frac{1}{5}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}=$

$\frac{1}{5}\begin{bmatrix} -2e^{-3} & e^2 \\ e^{-3} & 2e^2 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}=\boxed{\frac{1}{5}\begin{bmatrix} 4e^{-3}+e^2 & -2e^{-3}+2e^2 \\ -2e^{-3}+2e^2 & e^{-3}+4e^2 \end{bmatrix}}$

2. Solve the discrete dynamical system $A = \begin{bmatrix} 1.24 & -0.7 \\ .85 & 0.1 \end{bmatrix}, \vec{x_0} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. Find and plot 10 points along the trajectory starting at the indicated point. What is the long-term behavior of the system starting at any initial condition? Write your solution in terms of the eigenvectors and eigenvalues. (8 points)

$(1.24-\lambda)(.1-\lambda)+.595=0$

$.124-1.24\lambda-.1\lambda+\lambda^2+.595=0$

$\lambda^2-1.34\lambda+.719=0$

$\lambda=\frac{1.34 \pm \sqrt{1.7956-2.876}}{2}$

$\approx \frac{1.34 \pm 1.0394i}{2} \approx .67 \pm .5197i$

$|\lambda|=\sqrt{.4489+.27}=\sqrt{.71899}\approx .8479<1$

Collapses to origin

$\vec{x}_0=\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ $\vec{x}_1=\begin{bmatrix} 4.8 \\ 4.45 \end{bmatrix}$ $\vec{x}_2=\begin{bmatrix} 2.837 \\ 4.525 \end{bmatrix}$

$\vec{x}_3=\begin{bmatrix} .35038 \\ 2.86 \end{bmatrix}, \vec{x}_4=\begin{bmatrix} -1.57 \\ .584 \end{bmatrix}, \vec{x}_5=\begin{bmatrix} -2.356 \\ -1.226 \end{bmatrix},$

$\vec{x}_6=\begin{bmatrix} -2.028 \\ -2.13 \end{bmatrix}, \vec{x}_7=\begin{bmatrix} -1.02 \\ -1.937 \end{bmatrix}, \vec{x}_8=\begin{bmatrix} .0865 \\ -1.064 \end{bmatrix}$

$\vec{x}_9=\begin{bmatrix} .852 \\ -.033 \end{bmatrix}, \vec{x}_{10}=\begin{bmatrix} 1.079 \\ .721 \end{bmatrix}$

the solution is not real; needs Euler's formula

$\vec{x}_n=c_1\vec{v}_1\lambda_1^n+c_2\vec{v}_2\lambda_2^n$

3. The table below shows the revenue (in billions of dollars) for the Acme Motors Corporation from 2005 to 2010. Graph the points. Select a polynomial model for the data and find a regression equation of best fit (use the fewest variables needed to get a decent fit—if you're leading coefficients are nearly zero, you have too many variables). Use that equation to predict revenue for 2016 ($x = 16$). Does your prediction make sense? (8 points)

Year after 2000	**5**	**6**	**7**	**8**	**9**	**10**
Revenue (y)	21.2	24.1	27.2	29.3	32.0	32.5

linear is best

$$A = \begin{bmatrix} 1 & 5 \\ 1 & 6 \\ 1 & 7 \\ 1 & 8 \\ 1 & 9 \\ 1 & 10 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 21.2 \\ 24.1 \\ 27.2 \\ 29.3 \\ 32.0 \\ 32.5 \end{bmatrix}$$

$$(A^TA)^{-1}A^T\vec{y} = \begin{bmatrix} 10.1 \\ 2.35 \end{bmatrix} \quad y = \beta_0 + \beta_1 x$$

$$\vec{y} = 10.1 + 2.35x$$

$y(16) = 47.7$ billion

it's not unreasonable if company keeps growing

quadratic

$$A = \begin{bmatrix} 1 & 5 & 25 \\ 1 & 6 & 36 \\ 1 & 7 & 49 \\ 1 & 8 & 64 \\ 1 & 9 & 81 \\ 1 & 10 & 100 \end{bmatrix}$$

$$y = -2.87 + 5.99x - .24x^2$$

$$y(16) = 31.53$$

4. Prove or disprove that if λ is an eigenvalue of A, it is also an eigenvalue of A^2. (5 points)

$$A\vec{x} = \lambda\vec{x}$$

$$A^2x = A(\lambda\vec{x}) = \lambda(A\vec{x}) = \lambda\cdot(\lambda\vec{x}) = \lambda^2\vec{x}$$

false. λ^2 is an eigenvalue of A^2

5. Given the linear transformation defined by $A = \begin{bmatrix} 3 & -2 & 6 & -1 & 15 \\ 4 & 3 & -8 & 10 & -14 \\ 2 & -3 & 4 & -4 & 20 \\ 0 & 6 & 1 & 2 & 8 \end{bmatrix}$, determine if the transformation is any of the following. Explain your reasoning in each case. (3 points each)

a. One-to-one

rref $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 5.4 \\ 0 & 1 & 0 & 0 & 2.8 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4.4 \end{bmatrix}$

no

there is not a pivot in every column

b. Onto

yes, there is a pivot in every row

6. Consider a generic 49×43 matrix. Is it possible for the linear transformation defined by the matrix to be:

a. One-to-one? Why or why not? (3 points)

yes. there can be 43 pivots which would be one in every column

b. Onto? Why or why not? (3 points)

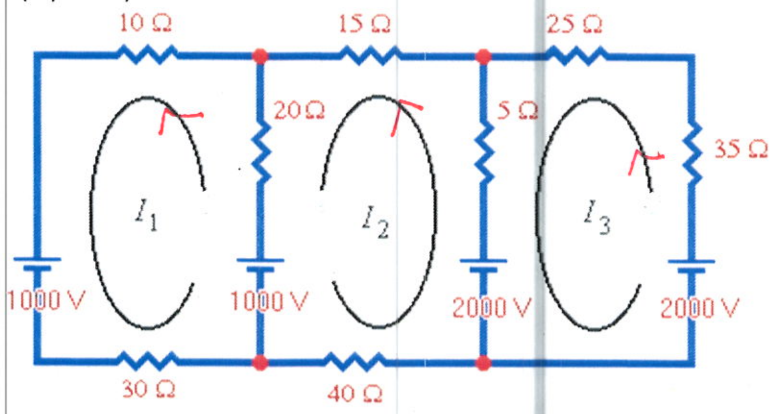
no. there cannot be 49 pivots since there are only 43 columns.

c. If the matrix has 43 pivots, what is the dimension of the kernel and the range? (4 points)

$\dim(\operatorname{Ker} A) = 0$

$\dim(\text{range}) = 43$

7. Set up and solve the loop circuit diagram below. Round your values for the currents to five significant digits. (8 points)



$$60I_1 - 20I_2 = 0$$
$$-5I_3 - 20I_1 + 80I_2 = 1000$$
$$-5I_2 + 65I_3 = 0$$

$$\left[\begin{array}{ccc|c} 60 & -20 & 0 & 0 \\ -20 & 80 & -5 & 1000 \\ 0 & -5 & 65 & 0 \end{array}\right]$$

$$\text{rref} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4.5694 \\ 0 & 1 & 0 & 13.7082 \\ 0 & 0 & 1 & 1.05448 \end{array}\right]$$

$$\vec{I} = \begin{bmatrix} 4.5694 \\ 13.708 \\ 1.0545 \end{bmatrix}$$

8. Encode the message WHAT WE THINK WE BECOME with the matrix $A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & -1 & -1 \\ 5 & 2 & 1 \end{bmatrix}$ using 0 as a space and the corresponding position (number) in the alphabet for letters. (8 points)

$$\begin{bmatrix} 23 & 8 & 1 \\ 20 & 0 & 23 \\ 5 & 0 & 20 \\ 8 & 9 & 14 \\ 11 & 0 & 23 \\ 5 & 0 & 2 \\ 5 & 3 & 15 \\ 13 & 5 & 0 \end{bmatrix}$$

73 40 16 195 86 43 120 50

25 75 35 13 159 68 34 30

14 7 86 37 17 37 21 8

9. Find the volume of the tetrahedron whose corner is defined by the vertices $(3,-1,1)$, $(4,-4,4)$, $(1,1,1)$, $(0,0,1)$. Sketch a graph of the shape. (4 points)

$$\frac{1}{6}\left|\det\left(\begin{bmatrix} 3 & -1 & 1 & 1 \\ 4 & -4 & 4 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}\right)\right| = \frac{1}{6}\left|-12\right| = 2$$

10. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 5 & -1 & -3 & 4 \\ 2 & 3 & 1 & 0 & 7 & 4 \\ 0 & -1 & 11 & 4 & -1 & 12 \\ 8 & 4 & -2 & 6 & 1 & -3 \end{bmatrix}$. Find a *basis* for:

a. The nullspace of A (3 points)

$$\text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -1339/166 & -649/166 \\ 0 & 1 & 0 & 0 & 1347/166 & 623/166 \\ 0 & 0 & 1 & 0 & -201/166 & 93/166 \\ 0 & 0 & 0 & 1 & 424/83 & 199/83 \end{bmatrix}$$

$x_1 = 1339/166 x_4 + 649/166 x_5$

$x_2 = -1347/166 x_4 - 623/166 x_5$

$x_3 = 201/166 x_4 - 93/166 x_5$

$x_4 = -424/83 \cancel{x_4} - 199/83 x_5$

$x_4 = \quad x_4 \quad x_5$

$x_5 =$

$$\text{Null}A = \left\{ \begin{bmatrix} 1339 \\ -1347 \\ 201 \\ -848 \\ 166 \\ 0 \end{bmatrix}, \begin{bmatrix} 649 \\ -623 \\ -93 \\ -398 \\ 0 \\ 166 \end{bmatrix} \right\}$$

b. The column space of A (3 points)

$$\text{Col}\,A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 11 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 6 \end{bmatrix} \right\}$$

c. The row space of A (3 points)

$$\text{row}A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \\ -1 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \\ 7 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 11 \\ 4 \\ 1 \\ 12 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \\ -2 \\ 6 \\ 1 \\ -3 \end{bmatrix} \right\}$$

d. The rank of A (2 points)

4

11. Find the change of basis matrix to transition from C to B for $B = \left\{ \begin{bmatrix} 9 \\ -3 \\ 15 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 2 \\ -3 \end{bmatrix} \right\}$, $C = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Use it to convert $[\vec{x}]_C = \begin{bmatrix} 1 \\ -3 \\ 4 \\ -2 \end{bmatrix}$ to $[\vec{x}]_B$. What is the transition matrix for $P_{B \leftarrow C}$? (7 points)

$$P_B [\vec{x}]_B = P_C [\vec{x}]_C$$

$$[x]_B = P_B^{-1} P_C [\vec{x}]_C$$

$$P_B = \begin{bmatrix} 9 & 3 & 0 & 3 \\ -3 & 0 & -5 & -4 \\ 15 & 0 & 6 & 2 \\ 4 & 1 & 8 & -3 \end{bmatrix}$$

$$P_B^{-1} P_C [x]_C = \begin{bmatrix} 637/969 \\ -2549/969 \\ -173/323 \\ -11/34 \end{bmatrix}$$

$$P_C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ -1 & 1 & 1 & 1 \end{bmatrix}$$

$$P_B^{-1} P_C = P_{B \leftarrow C} = \begin{bmatrix} 166/969 & -1/323 & 45/323 & 13/323 \\ -179/1938 & -16/323 & -175/646 & 93/646 \\ -75/323 & 50/323 & 11/323 & -4/323 \\ 3/34 & 1/17 & -5/34 & -9/34 \end{bmatrix}$$

12. Solve the system $\begin{cases} 4x_1 - x_2 + x_3 = -8 \\ x_1 + x_2 + 3x_3 = 10 \\ 5x_1 - 2x_2 + 2x_3 = 14 \end{cases}$ by any method. (7 points)

$$\left[\begin{array}{ccc|c} 4 & -1 & 1 & -8 \\ 1 & 1 & 3 & 10 \\ 5 & -2 & 2 & 14 \end{array}\right] \Rightarrow \text{rref} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & -19 \\ 0 & 0 & 1 & 13 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} -10 \\ -19 \\ 13 \end{bmatrix}$$