MTH 265, Exam #2, Part I, Fall 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. Use technology to graph each of the vector field equations. Attach the graphs. On each graph note any curves where one component is zero (horizontal or vertical vectors: these are called nullclines). Determine if either of the fields is conservative. (20 points)
   a. $\vec{F}(x,y) = \sin x\,\hat{\imath} + \cos y\,\hat{\jmath}$

   $$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin x & \cos y & 0 \end{vmatrix} = 0\hat{\imath} - 0\hat{\jmath} + 0\hat{k}$$ conservative

   b. $\vec{F}(x,y) = 3x\hat{\imath} - 2y\hat{\jmath}$

   $$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x & -2y & 0 \end{vmatrix} = 0\hat{\imath} - 0\hat{\jmath} + 0\hat{k}$$ conservative

2. Find the volume bounded by the coordinate axes and the plane $3x + 6y + z = 12$. (12 points)

$$\int_0^4 \int_0^{2-\frac{1}{2}x} 12 - 3x - 6y \, dy\, dx$$

$$\int_0^4 12y - 3xy - 3y^2 \Big|_0^{2-\frac{1}{2}x} dx =$$

$$\int_0^4 \frac{3}{4}x^2 - 6x + 12\, dx =$$

$$\frac{1}{4}x^3 - 3x^2 + 12x \Big|_0^4 = 16$$

$z = 12 - 3x - 6y$

$\frac{3x + 6y = 12}{6}$

$\frac{1}{2}x + y = 2$

$y = 2 - \frac{1}{2}x$

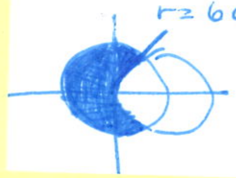
$3x = 12 \Rightarrow x = 4$

3. Set up a double integral to find the area inside the circle $x^2 + y^2 = 9$ and outside $x^2 + y^2 = 6x$. Sketch the graph. Evaluate your integral. (8 points)

$r = 3$

$r^2 = 6r\cos\theta$

$r = 6\cos\theta$

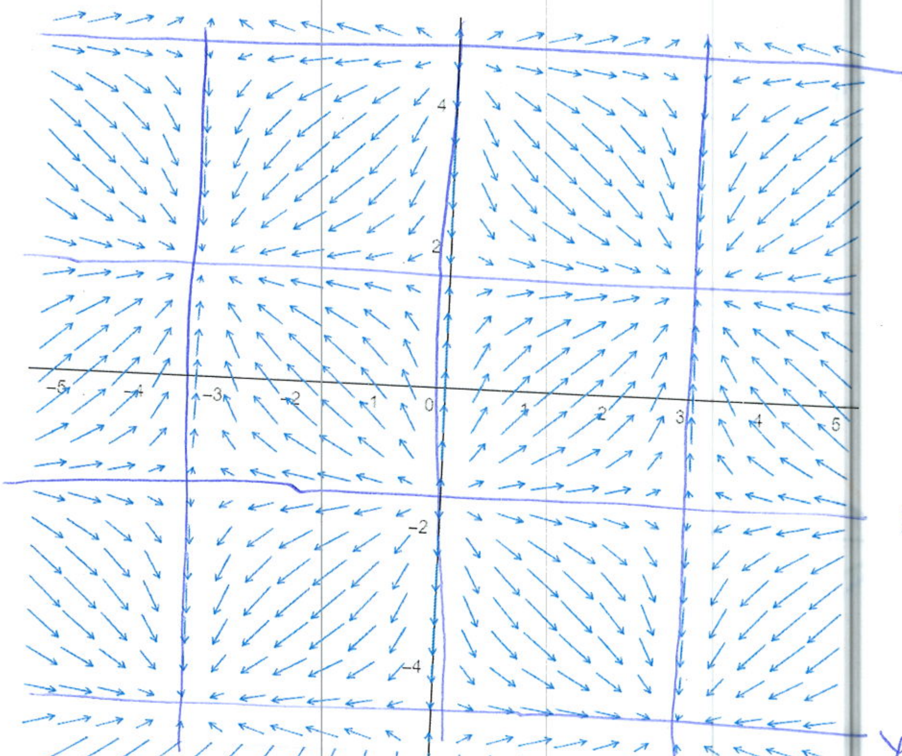


$3 = 6\cos\theta$

$\frac{1}{2} = \cos\theta$

$\theta = \pi/3$

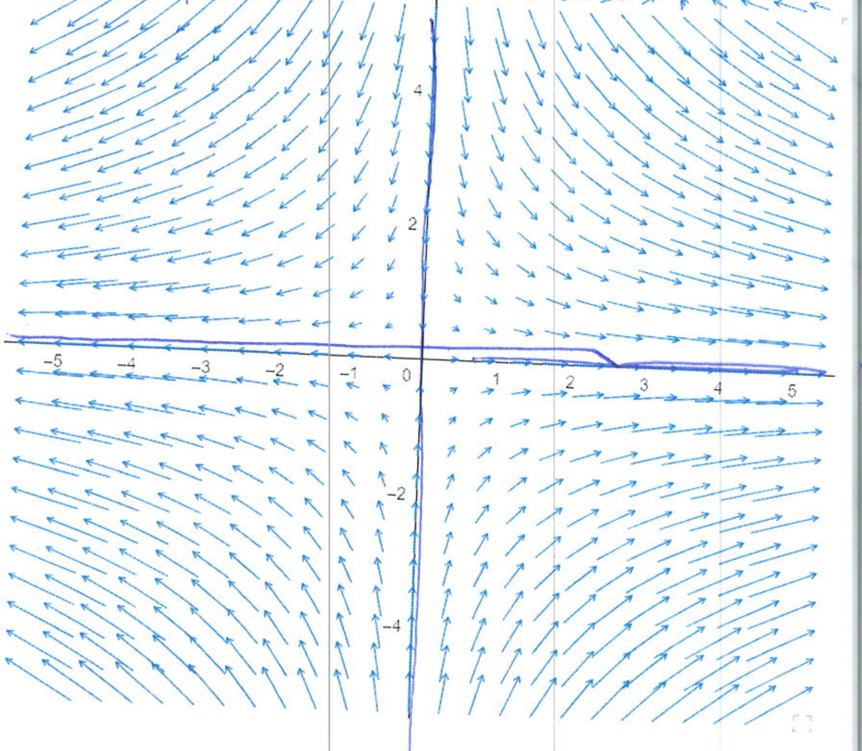
See attached



1a.

$x = \pi/2, -\pi/2, 3\pi/2\ -3\pi/2$, etc

$y = 0, \pi, -\pi, 2\pi, -2\pi$, etc.

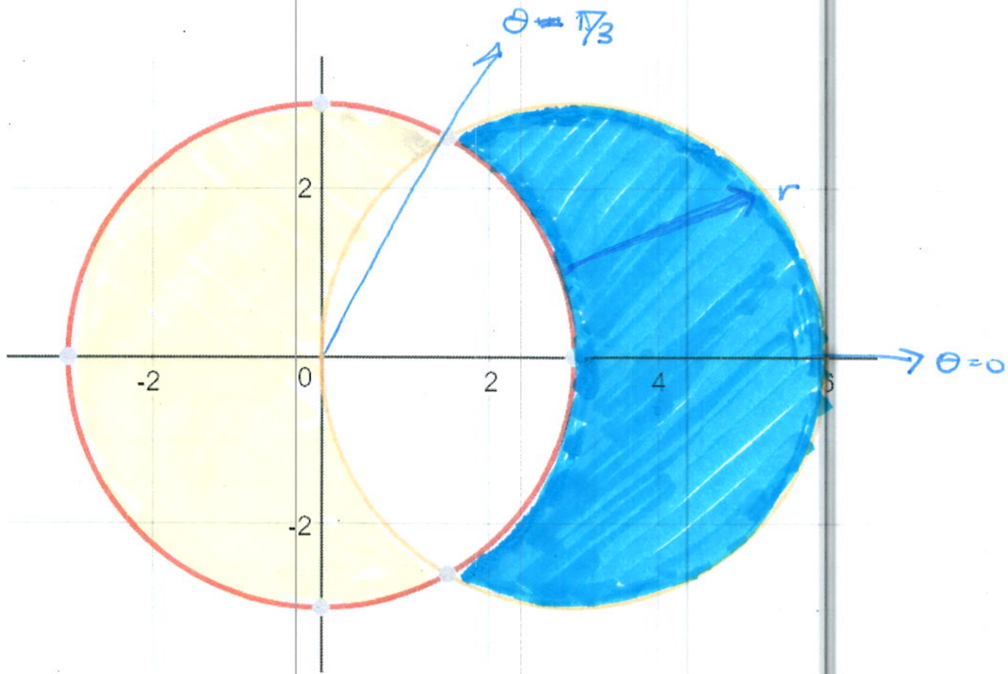


1b

$y = 0.$

$x = 0$

#3



these two regions are equal
and the blue region is easier to calculate

using symmetry

$$A = 2\int_0^{\pi/3}\int_3^{6\cos\theta} r\,dr\,d\theta = \int_0^{\pi/3} r^2\Big|_3^{6\cos\theta} d\theta =$$

$$\int_0^{\pi/3} 36\cos^2\theta - 9\,d\theta = \int_0^{\pi/3} 18 + 18\cos 2\theta - 9\,d\theta =$$

$$\int_0^{\pi/3} 9 + 18\cos 2\theta\,d\theta = 9\theta + 9\sin 2\theta\Big|_0^{\pi/3} =$$

$$9(\pi/3) + 9\left(\frac{\sqrt{3}}{2}\right) - 0 - 0 = \boxed{3\pi + \frac{9\sqrt{3}}{2}}$$

4. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F}(x,y,z) = x^2y\hat{\imath} + xy^2\hat{\jmath} + z^2\hat{k}$ on the path $C: \vec{r}(t) = \frac{1}{2}\sin t\,\hat{\imath} + \frac{1}{3}\cos t\,\hat{\jmath} + t^3\hat{k}$ on $[0, \frac{\pi}{2}]$. (10 points)

$F(t) = \frac{1}{4}\sin^2 t\,\frac{1}{3}\cos t\,\hat{\imath} + \frac{1}{2}\sin t \cdot \frac{1}{9}\cos^2 t\,\hat{\jmath} + t^6\hat{k}$

$r'(t) = \frac{1}{2}\cos t\,\hat{\imath} - \frac{1}{3}\sin t\,\hat{\jmath} + 3t^2\hat{k}$

$\vec{F}\cdot d\vec{r} = \frac{1}{24}\sin^2 t\cos^2 t - \frac{1}{54}\sin^2 t\cos^2 t + 3t^8$

$= \frac{5}{216}\sin^2 t\cos^2 t + 3t^8 = \frac{5}{216}\,\frac{1}{2}(1-\cos 2t)(\frac{1}{2})(1+\cos 2t) + 3t^8 =$

$\frac{5}{864}(1-\cos^2 2t) + 3t^8 = \frac{5}{864}\left(1 - \frac{1}{2}(1+\cos 4t)\right) + 3t^8 = \frac{5}{864}\left(\frac{1}{2} - \frac{1}{2}\cos 4t\right) + 3t^8$

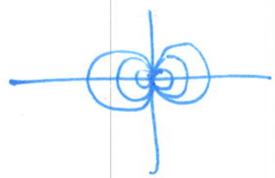
$\int_0^{\pi/2} \frac{5}{864}\left(\frac{1}{2} - \frac{1}{2}\cos 4t\right) + 3t^8\,dt = \frac{5}{864}\left(\frac{1}{2}t - \frac{1}{8}\sin 4t\right) + \frac{1}{3}t^9 \Big|_0^{\pi/2} =$

$\frac{5}{864}\left(\frac{\pi}{4} - 0\right) + \frac{\pi^9}{1536} = \frac{5\pi}{3456} + \frac{\pi^9}{1536}$

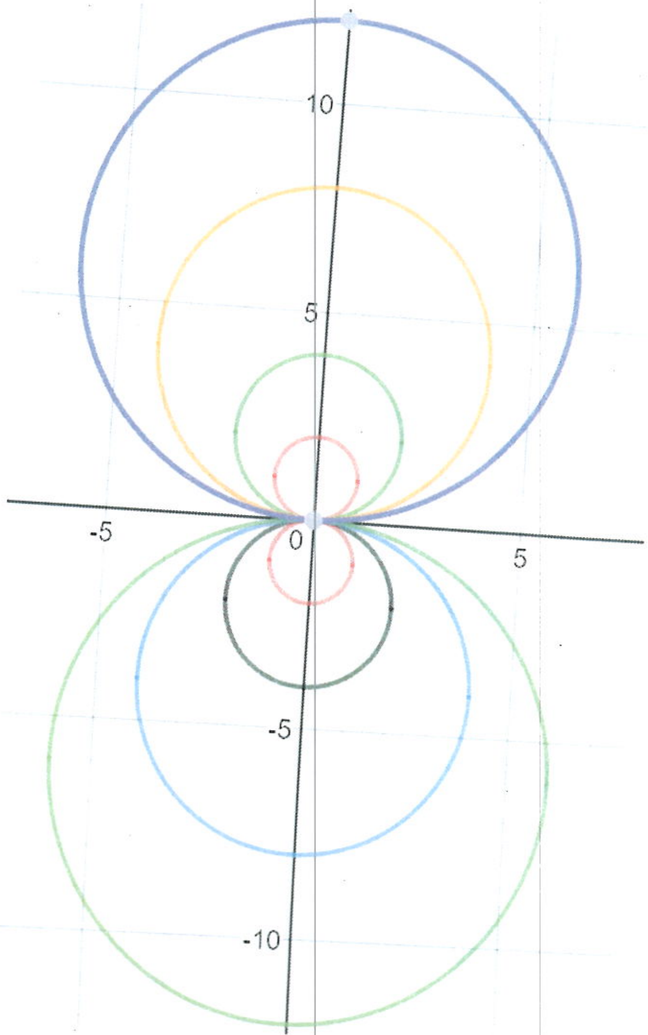
5. Sketch the level curves for $f(x,y) = \frac{4y}{x^2+y^2}$ for values of z in $[-2,2]$. Use technology to create a graph of the curve, and to verify your level curves. Explain in your own words how the two graphs are related. (10 points)

$z = \frac{4y}{x^2+y^2}$

$x^2+y^2 = \frac{4y}{z}$



explanations may vary but should include references to crossections in z, at different values, evaluate and sketch resulting graph.



10
5
-5
0
5
-5
-10

MTH 265, Exam #2, Part II, Fall 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

6. Convert the triple integrals to the given coordinate systems and then complete the integration. Describe the region being integrated (over). (10 points each)

   a. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2-1}^{1-x^2-y^2} x\,dz\,dy\,dx$ in cylindrical.

$1-r^2$

r^2-1

$$\int_0^{2\pi}\int_0^1\int_{r^2-1}^{1-r^2} r\cos\theta\, r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r^2\cos\theta\, z\Big|_{r^2-1}^{1-r^2} dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^1 r^2\cos\theta\left[(1-r^2)-(r^2-1)\right]dr\,d\theta = \int_0^{2\pi}\int_0^1 r^2\cos\theta\,(2-2r^2)\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^1 (2r^2-2r^4)\cos\theta\,dr\,d\theta = \int_0^{2\pi}\left(\tfrac{2}{3}r^3-\tfrac{2}{5}r^5\right)\Big|_0^1\cos\theta\,d\theta =$$

$$\frac{4}{15}\int_0^{2\pi}\cos\theta\,d\theta = \frac{4}{15}\sin\theta\Big|_0^{2\pi} = 0$$

   b. $\int_{-5}^{5}\int_0^{\sqrt{25-x^2}}\int_0^{\sqrt{25-x^2-y^2}} \frac{e^{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}\,dz\,dy\,dx$ in spherical.

plane

$$\int_0^{\pi/2}\int_0^{\pi}\int_0^5 \frac{e^\rho}{\cancel{\rho^2}}\cancel{\rho^2}\sin\varphi\,d\rho\,d\theta\,d\varphi = \int_0^{\pi/2}\int_0^{\pi} e^\rho\Big|_0^5\sin\varphi\,d\theta\,d\varphi =$$

$$(e^5-1)\pi\int_0^{\pi/2}\sin\varphi\,d\varphi = (e^5-1)\pi\left(-\cos\varphi\Big|_0^{\pi/2}\right) = (e^5-1)\pi(-0+1)$$

$$= (e^5-1)\pi$$

7. Set up a double integral to find the volume under $f(x,y) = \frac{\ln(x^2+y^2)}{\sqrt{x^2+y^2}}$ on the region $x^2+y^2 \le 16, x \ge 0$. Do not integrate. (8 points)

$f(r,\theta) = \frac{\ln r^2}{r} = \frac{2\ln r}{r}$

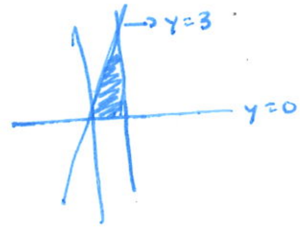
$$\int_{-\pi/2}^{\pi/2}\int_0^4 \frac{2\ln r}{\cancel{r}}\,\cancel{r}\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^4 2\ln r\,dr\,d\theta$$

8. Change the order of integration in $\int_0^3 \int_{y/3}^1 \frac{y^2}{1+x^4}\,dx\,dy$ so that it can be integrated. Then complete the integration. (10 points)

$x = y/3$

$y = 3x$

$x = 1$



$$\int_0^1\int_0^{3x} \frac{y^2}{1+x^4}\,dy\,dx = \int_0^1 \frac{1}{3}y^3 \cdot \frac{1}{1+x^4}\Big|_0^{3x} dx = \int_0^1 \frac{9x^3}{1+x^4}\,dx$$

$u = 1+x^4$

$du = 4x^3\,dx$

$\frac{1}{4}du = x^3\,dx$

$\frac{9}{4}\int \frac{1}{u}\,du$

$$\frac{9}{4}\ln|1+x^4|\Big|_0^1 =$$

$$\frac{9}{4}\left[\ln 2 - \cancel{\ln 1}\right] = \frac{9\ln 2}{4}$$

9. Determine if the vector field $\vec{F}(x,y,z) = yz\cos x\,\hat{\imath} + (2y + z\sin x)\hat{\jmath} + (y\sin x - 1)\hat{k}$ is conservative. Find the potential function. (10 points)

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz\cos x & 2y+z\sin y & y\sin x - 1 \end{vmatrix} = (\sin y - \sin x)\hat{\imath} - (y\cos x - y\cos x)\hat{\jmath} + (z\cos x - z\cos x)\hat{k} = \vec{0}$$

conservative

$\int yz\cos x\,dx = yz\sin x + f(y,z)$

$\int 2y + z\sin x\,dy = y^2 + yz\sin x + g(x,z)$

$\int y\sin x - 1\,dz = yz\sin x - z + h(x,y)$

$\varphi = yz\sin x + y^2 - z + K$

10. Evaluate the line integral $\int_C 2xyzds$ on the path $\vec{r}(t) = t^2\hat{\imath} + 3t\hat{\jmath} + \sqrt{t}\hat{k}$ on $[0,1]$. (8 points)

$$\int_0^1 2\cdot t^2 \cdot 3t \cdot t^{1/2}\, ds$$

$$r'(t) = 2t\hat{\imath} + 3\hat{\jmath} + \frac{1}{2\sqrt{t}}\hat{k}$$

$$\|r'(t)\| = \sqrt{4t^2 + 9 + \frac{1}{4t}} = \sqrt{\frac{16t^3+36t+1}{4t}} = \frac{\sqrt{16t^3+36t+1}}{2t^{1/2}}$$

$$= \int_0^1 \cancel{6}^{3} t^{\cancel{7/2}\,3} \cdot \frac{\sqrt{16t^3+36t+1}}{\cancel{2}t^{\cancel{1/2}}}\, dt =$$

$$\int_0^1 3t^3\sqrt{16t^3+36t+1}\, dt$$

integrate numerically

$$\approx 4.6335$$

11. Find ∇f and $\nabla^2 f$ for $f(x,y,z) = 2x^2 z\tan y + \sqrt[3]{z}$. (8 points)

$z^{1/3}$

$$\nabla f = \langle 4xz\tan y,\ 2x^2 z\sec^2 y,\ 2x^2\tan y + \tfrac{1}{3}z^{-2/3} \rangle$$

$$\nabla f^2 = 4z\tan y + 4x^2 z\sec^2 y\tan y - \tfrac{2}{9}x^{-5/3}$$

12. Find $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ for $\vec{F}(x,y,z) = \sin(xy)\hat{\imath} + \csc(yz)\,\hat{\jmath} + \frac{x}{z}\hat{k}$. (10 points)

$$\vec{\nabla}\cdot\vec{F} = y\cos xy - z\csc yz\cot yz - \frac{x}{z^2}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin xy & \csc yz & \frac{x}{z} \end{vmatrix} = (0 + y\csc yz\cot yz)\hat{\imath} - \left(\tfrac{1}{z} - 0\right)\hat{\jmath} + (0-0)\hat{k}$$

$$= y\csc(yz)\cot(yz)\hat{\imath} - \tfrac{1}{z}\hat{\jmath} + 0\hat{k}$$

Cylindrical

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$x^2 + y^2 = r^2$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \theta$$

Spherical

$$x = \rho\cos\theta\sin\phi$$
$$y = \rho\sin\theta\sin\phi$$
$$z = \rho\cos\phi$$
$$x^2 + y^2 + z^2 = \rho^2$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \theta$$
$$\cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \phi$$
$$x^2 + y^2 = \rho^2\sin^2\phi = r^2$$

Dels

$$\frac{\partial}{\partial x} = \text{ partial derivative with respect to } x$$
$$\nabla f = grad\, f$$
$$\nabla^2 f = \nabla \cdot (\nabla f) = Laplacian\ of\ f$$
$$\nabla \cdot \vec{F} = div\, \vec{F}$$
$$\nabla \times \vec{F} = curl\, \vec{F}$$

Misc

$$ds = \|\vec{r}\,'(t)\|dt$$