MTH 266, Final Exam, Part I, Fall 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available. On this portion of the exam, you may **NOT** use a calculator.

1. Compute $\begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{bmatrix}$. (10 points)

$$= \begin{bmatrix} 1(3)+0(-1)+3(6) & 1(0)+0(4)+3(5) \\ 2(3)-5(-1)+4(6) & 2(0)-5(4)+4(5) \end{bmatrix} =$$

$$\begin{bmatrix} 21 & 15 \\ 35 & 0 \end{bmatrix}$$

2. Compute $A + 3B$ given $A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 6 \\ 2 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 0 \end{bmatrix}$ (10 points)

$$\begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 6 \\ 2 & 1 & -1 \end{bmatrix} + \begin{bmatrix} -3 & 3 & 3 \\ 0 & 6 & 6 \\ 3 & 3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 7 \\ 3 & 7 & 12 \\ 5 & 4 & -1 \end{bmatrix}$$

3. Find the determinant by any means. $\begin{vmatrix} 1 & -1 & 11 \\ 3 & 4 & -3 \\ 8 & -2 & 0 \end{vmatrix}$ (15 points)

$$11\begin{vmatrix} 3 & 4 \\ 8 & -2 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 8 & -2 \end{vmatrix} = 11(-6-32)+3(-2+8)$$

$$= 11(-38)+3(6) = -400$$

4. Given the system of equations $\begin{cases} x_1 + 2x_2 - 3x_3 = -3 \\ -x_1 - 2x_2 - x_3 = 4 \\ -3x_2 - 7x_3 = 10 \end{cases}$, write the *system* as:

a. An augmented matrix (5 points)

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -3 \\ -1 & -2 & -1 & 4 \\ 0 & -3 & -7 & 10 \end{array}\right]$$

b. A vector equation (5 points)

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix} x_2 + \begin{bmatrix} -3 \\ -1 \\ -7 \end{bmatrix} x_3 = \begin{bmatrix} -3 \\ 4 \\ 10 \end{bmatrix}$$

c. A matrix equation. (5 points)

$$\begin{bmatrix} 1 & 2 & -3 \\ -1 & -2 & -1 \\ 0 & -3 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 10 \end{bmatrix}$$

d. Solve the system using the augmented matrix and row operations. State whether the solution of the system is consistent or inconsistent. If the system is consistent, state whether it is independent or dependent. Write an independent solution in vector form; write a dependent solution in parametric form. (15 points)

$R_1 + R_2 \to R_2$ $\left[\begin{array}{ccc|c} 1 & 2 & -3 & -3 \\ 0 & 0 & -4 & 1 \\ 0 & -3 & -7 & 10 \end{array}\right]$ $R_2 \leftrightarrow R_3$ $\left[\begin{array}{ccc|c} 1 & 2 & -3 & -3 \\ 0 & -3 & -7 & 10 \\ 0 & 0 & -4 & 1 \end{array}\right]$

$-\frac{1}{3} R_2 \to R_2$
$-\frac{1}{4} R_3 \to R_3$ $\left[\begin{array}{ccc|c} 1 & 2 & -3 & -3 \\ 0 & 1 & 7/3 & -10/3 \\ 0 & 0 & 1 & -1/4 \end{array}\right]$ $-7/3 R_3 + R_2 \to R_2$
$3 R_3 + R_1 \to R_1$ $\left[\begin{array}{ccc|c} 1 & 2 & 0 & -15/4 \\ 0 & 1 & 0 & -11/4 \\ 0 & 0 & 1 & -1/4 \end{array}\right]$

$-2 R_2 + R_1 \to R_1$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7/4 \\ 0 & 1 & 0 & -11/4 \\ 0 & 0 & 1 & -1/4 \end{array}\right]$

$$\vec{x} = \begin{bmatrix} 7/4 \\ -11/4 \\ -1/4 \end{bmatrix}$$

5. Find the inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (8 points)

$$A^{-1} = \frac{1}{4-6}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

6. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$. Be sure to clearly indicate the characteristic equation, and which eigenvalues and eigenvectors go together. (20 points)

$(4-\lambda)(1-\lambda)+2=0$

$\lambda^2-5\lambda+4+2=0$

$\lambda^2-5\lambda+6=0$

$(\lambda-3)(\lambda-2)=0$

$\lambda=3,2$

$\lambda_1=3$

$\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$ $x_1=2x_2$ $\vec{v}_1=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2=2$

$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$ $x_1=x_2$ $\vec{v}_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

7. Given the vectors $\mathbf{u} = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$ find the following.

a. $\mathbf{u} \cdot \mathbf{v}$ (5 points)

$-2(1) + 5(-3) + 0(2) = -2 - 15 = -17$

b. The distance between **u** and **v**. (7 points)

$u - v = \begin{bmatrix} -3 \\ 8 \\ -2 \end{bmatrix}$ $\|u - v\| = \sqrt{9 + 64 + 4} = \sqrt{77}$

c. A unit vector in the direction of **v**. (5 points)

$\|v\| = \sqrt{1 + 9 + 4} = \sqrt{14}$

$\hat{v} = \begin{bmatrix} 1/\sqrt{14} \\ -3/\sqrt{14} \\ 2/\sqrt{14} \end{bmatrix}$

d. Are **u** and **v** orthogonal? Why or why not? (5 points)

no, since the dot product is not 0

8. Given that A and B are 4×4 matrices with det $A = 2$ and det $B = -8$, find the following. (4 points each)

a) $det\ AB$ -16

b) $det\ A^{-1}$ $\frac{1}{2}$

c) $det\ 3A$ $3^4(2) = 81(2) = 162$

9. Find the closest point to $\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$ in the subspace W spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 2 \\ 0 \\ 3 \end{bmatrix}$.

(15 points)

$$\text{proj}_{v_1} \vec{y} = \frac{3(1)-1(-1)+1(-2)+13(2)}{1^2+(-1)^2+(-2)^2+2^2} \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix} = \frac{28}{10} \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 14/5 \\ -14/5 \\ -28/5 \\ 28/5 \end{bmatrix}$$

$$\text{proj}_{v_2} \vec{y} = \frac{3(-4)-1(2)+1(0)+13(3)}{(-4)^2+2^2+0^2+9} \begin{bmatrix} -4 \\ 2 \\ 0 \\ 3 \end{bmatrix} = \frac{25}{29} \begin{bmatrix} -4 \\ 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -100/29 \\ 50/29 \\ 0 \\ 75/29 \end{bmatrix}$$

$$\vec{y}_{\parallel} = \begin{bmatrix} 14/5 \\ -14/5 \\ -28/5 \\ 28/5 \end{bmatrix} + \begin{bmatrix} -100/29 \\ 50/29 \\ 0 \\ 75/29 \end{bmatrix} = \begin{bmatrix} -94/145 \\ -156/145 \\ -28/5 \\ 1187/145 \end{bmatrix}$$

10. Given $\mathbf{u}_1 = \begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix}$, *and* $\mathbf{u}_2 = \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix}$ and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Determine if $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for W. If it is, make it an orthonormal basis. (15 points)

$$u_1 \cdot u_2 = 5(-4)-4(1)+0(-3)+3(8) =$$
$$-20-4+0+24=0$$

yes, it is an orthogonal basis

$$\|u_1\| = \sqrt{25+16+9} = \sqrt{50} = 5\sqrt{2}$$

$$\|u_2\| = \sqrt{16+1+9+64} = \sqrt{90} = 3\sqrt{10}$$

$$\left\{ \begin{bmatrix} 5/5\sqrt{2} \\ -4/5\sqrt{2} \\ 0 \\ 3/5\sqrt{2} \end{bmatrix}, \begin{bmatrix} -4/3\sqrt{10} \\ 1/3\sqrt{10} \\ -3/3\sqrt{10} \\ 8/3\sqrt{10} \end{bmatrix} \right\}$$

11. Given the basis of W in question #10, and the vector $\vec{y} = \begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ decompose this vector into $\vec{y} = \vec{y_{\parallel}} + \vec{y_{\perp}}$ with $\vec{y_{\parallel}} = proj_W \vec{y}$. (15 points)

$$proj_{u_1} \vec{y} = \frac{5(5)+2(-4)+1(0)+0(3)}{25+16+0+9} \begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix} = \frac{17}{50} \begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 17/10 \\ -34/25 \\ 0 \\ 51/50 \end{bmatrix}$$

$$proj_{u_2} \vec{y} = \frac{5(-4)+2(1)+1(-3)+0(8)}{16+1+9+64} \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix} = \frac{-21}{90} \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix} = \begin{bmatrix} -14/15 \\ -7/30 \\ 7/10 \\ -28/15 \end{bmatrix}$$

$$\vec{y_{\parallel}} = \begin{bmatrix} 17/10 \\ -34/25 \\ 0 \\ 51/50 \end{bmatrix} + \begin{bmatrix} -14/15 \\ -7/30 \\ 7/10 \\ -28/15 \end{bmatrix} = \begin{bmatrix} 23/30 \\ -239/150 \\ 7/10 \\ -127/150 \end{bmatrix}$$

$$\vec{y_{\perp}} = \vec{y} - \vec{y_{\parallel}} = \begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 23/30 \\ -239/150 \\ 7/10 \\ -127/150 \end{bmatrix} = \begin{bmatrix} 127/30 \\ 539/150 \\ 3/10 \\ 127/150 \end{bmatrix}$$

MTH 266, Final Exam, Part II, Fall 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available. On this portion of the exam, you *may* use a calculator to perform elementary matrix operations. Support your answers with work (reproduce the reduced matrices from your calculator) or other justification for full credit.

12. Determine if each statement is True or False. (3 points each)

a. T F The homogeneous system $A\mathbf{x} = \mathbf{0}$ is always consistent.

b. T F If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent, then so is $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$.

c. T F Two eigenvectors corresponding to the same eigenvalue are always linearly dependent.

only if eigenvalues are not repeated

d. T F Matrix multiplication is commutative.

e. T F If $\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ then **u** and **v** are orthogonal.

f. T F If a system of equations has a free variable then it has a unique solution.

g. T F If A is a $n \times n$ matrix, then A is invertible.

h. T F If two vectors are orthogonal, they are linearly independent.

i. T F If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, then **u**, **v**, and **w** are not in R^2.

j. T F If det A is zero, then two rows or two columns of A are the same, or a row or a column is zero.

not necessarily - can be linear combo

k. T F If A and B are row equivalent, then their column spaces are the same.

row space yes

l. T F The vector space P_3 and R^3 are isomorphic.

P_3 iso to $\mathbb{R}^4$

m. T F An $n \times n$ matrix can have more than n eigenvalues.

n. T F If $\vec{y}$ is a linear combination of nonzero vectors from an orthogonal set, then the weights in the linear combination can be computed without row operations on a matrix.

o. T F If the columns of A are linearly independent, then the equation $A\vec{x} = \vec{b}$ has exactly one least-squares solution.

p. T F A least-squares solution of $A\vec{x} = \vec{b}$ is the point in the column space of A closest to $\vec{b}$.

13. Find a least squares solution of $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ by constructing the normal equations for $\vec{z}$ and solving for $\vec{z}$. (12 points)

$$(A^TA)^{-1}A^T\vec{b} = \vec{x}$$

$$A^T = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix} \quad A^TA = \begin{bmatrix} 6 & 2 \\ 2 & 11 \end{bmatrix}$$

$$(A^TA)^{-1} = \begin{bmatrix} 11/62 & -1/31 \\ -1/31 & 3/31 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

$$(A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 11/62 & -1/31 \\ -1/31 & 3/31 \end{bmatrix}\begin{bmatrix} 7 \\ 14 \end{bmatrix} = \begin{bmatrix} 49/62 \\ 35/31 \end{bmatrix} \approx \begin{bmatrix} .79 \\ 1.13 \end{bmatrix}$$

14. Given the basis $\{1, t, 1 - 3t^2\}$, find the representation of $p(t) = 3t^2 + 2t - 5$ in this basis. (10 points)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} = P_B \quad \vec{x} = \begin{bmatrix} -5 \\ 2 \\ 3 \end{bmatrix}$$

$$P_B^{-1}\vec{x} = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}\begin{bmatrix} -5 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ -1 \end{bmatrix} = [p(t)]_B$$

15. Show that the matrix $A = \begin{bmatrix} 1 & -3 \\ 4 & 5 \end{bmatrix}$ satisfies the conditions of a linear transformation. Use the generic vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, where the entries u_1, u_2, v_1, v_2 and the scalar c are real numbers. (20 points)

(1) $T(u+v) = T(u) + T(v)$

$$\begin{bmatrix} 1 & -3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix} = \begin{bmatrix} u_1+v_1-3u_2-3v_2 \\ 4u_1+4v_1+5u_2+5v_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 1 & -3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1-3u_2 \\ 4u_1+5u_2 \end{bmatrix} + \begin{bmatrix} v_1-3v_2 \\ 4v_1+5v_2 \end{bmatrix} = \begin{bmatrix} u_1+v_1-3u_2-3v_2 \\ 4u_1+4v_1+5u_2+5v_2 \end{bmatrix}$$

these are equal

(2) $T(c\vec{u}) = cT(\vec{u})$

$$\begin{bmatrix} 1 & -3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} = \begin{bmatrix} cu_1-3cu_2 \\ 4cu_1+5cu_2 \end{bmatrix}$$

$$c\begin{bmatrix} 1 & -3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c\begin{bmatrix} u_1-3u_2 \\ 4u_1+5u_2 \end{bmatrix} = \begin{bmatrix} cu_1-3cu_2 \\ 4cu_1+5cu_2 \end{bmatrix}$$

They are equal

(3) $T(\vec{0}) = \vec{0}$

$$\begin{bmatrix} 1 & -3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

thus, the transformation A is linear

16. Define the term *span*. Be as precise as possible in your definition. Use examples when this will help clarify a meaning, but do not only use examples in your definition. (6 points)

The span is the set of vectors that can be formed by the linear combination of a particular subset of vectors.

e.g. if $W = \text{span}\{\vec{u}_1, \vec{u}_2, \dots u_n\}$ then W is the set of any vector $\vec{v}$ that can be written as $k_1[\vec{u}_1] + k_2[u_2] + \cdots k_n[u_n] = \vec{v}$

17. List at least 10 properties of Invertible Matrices from the Invertible Matrix Theorem. If you can list all 20, you'll earn one point for each correct one. (10+ points)

if A is $n \times n$: Your answers will vary but can include:

1) the determinant is not zero
2) no eigenvalue of A is zero
3) the columns of A are linearly independent
4) the rows of A span $\mathbb{R}^n$
5) A is row equivalent to the identity
6) A is invertible
7) $A\vec{x} = \vec{0}$ has only the trivial solution
8) $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^n$
9) the columns of A form a basis for $\mathbb{R}^n$
10) A is one-~~to~~-one