MTH 266, Homework #5, Fall 2018 Name ____

Instructions: Write your work up neatly and attach to this page. Use exact values unless specifically asked to round. Show all work.

1. For each statement below determine if it is true or false. If the statement is false, briefly explain why it is false and give the true statement. Assume $\mathcal{B}$ is a basis for the vector space V.

   a. If $\vec{x}$ is in V and if $\mathcal{B}$ contains n vectors, then the B-coordinate vector of $\vec{x}$ is in R^n.

   b. The vector space $\mathbb{P}_3$ and R^3 are isomorphic.

   c. If $\mathcal{B}$ is the standard basis for R^n, then the $\mathcal{B}$-coordinate vector of an $\vec{x}$ in R^n is $\vec{x}$ itself.

   d. In some cases, a plane in R^3 can be isomorphic to R^2.

   e. The row space of A is the same as the column space of A^T.

   f. The sum of the dimensions of the row space and the null space of A equals the number of rows of A.

   g. The dimensions of the null space of A is the number of columns of A that are not pivot columns.

   h. If A and B are row equivalent, then their row spaces are the same.

   i. Dim Row A + dim Nul A = n

   j. If the equation $A\vec{x} = \vec{0}$ has only the trivial solution, then A is row equivalent to the *n*x*n* identity matrix.

   k. If the columns of A are linearly independent then the columns of A span R^n.

   l. If there is a $\vec{b}$ in R^n such that the equation $A\vec{x} = \vec{b}$ is consistent, then the solution is unique.

   m. If P_B is the change-of-coordinates matrix, then $\left[\vec{x}\right]_B = P_B\vec{x}$ for $\vec{x}$ in V.

   n. The columns of the change-of-coordinate matrix $\underset{C \leftarrow B}{P}$ are B-coordinate vectors of the vectors in C.

   o. The columns of $\underset{C \leftarrow B}{P}$ are linearly independent.

2. Answer the following questions, and then explain why you know this to be the case. State a theorem or definition that applies.
   a. If a 7x5 matrix A has rank 2, find dim Nul A, dim Row A, and rank A^T.

   b. Suppose a 6x8 matrix A has 4 pivot columns. What is dim Nul A? Is Col A = R^4? Why or why not?

   c. If the null space of an 8x7 matrix is 5-dimensional, what is the dimension of the Col space of A?

   d. If A is a 5x4 matrix, what is the largest possible dimension of the row space of A?

   e. If A is a 7x5 matrix, what is the smallest possible dimension of Nul A?

   f. Suppose the solutions of a homogeneous system of 5 linear equations in 6 unknowns are all multiples of one nonzero solution. Will the system necessarily have a solution for every possible choice of constants on the right sides of the equations? Explain.

3. Find a basis for the subspace and state the dimension.
   a. $\left\{ \begin{bmatrix} p-2q \\ 2p-5r \\ -2q+2r \\ -3p+6r \end{bmatrix} : p,q,r \in R \right\}$

   b. $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 5 \end{bmatrix} \right\}$

   c. $\begin{bmatrix} 1 & 2 & -4 & 3 & -2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 7 \\ 0 & 0 & 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, for Col A and Nul A.

   d. $\{1, 1-t, 2-4t+t^2, 6-18t+9t^2-t^3\}$

4. If $A = \begin{bmatrix} 1 & 1 & -2 & -4 & 1 & -2 \\ 1 & 2 & -3 & 0 & -2 & -3 \\ 1 & -1 & 0 & 0 & 1 & 6 \\ 1 & -2 & 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & -4 & 2 & -1 \end{bmatrix}$ is row equivalent to $B = \begin{bmatrix} 1 & 1 & -2 & -4 & 1 & -2 \\ 0 & 1 & -1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 & -13 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, find a basis for Col A, Row A and Nul A. Find rank A and dim Nul A without calculations.

5. Given the bases B and C, and the given vector in one the bases, find the coordinate vector in the other basis.
   a. $\mathcal{B}=\left\{\begin{bmatrix}1\\2\\4\end{bmatrix},\begin{bmatrix}2\\1\\3\end{bmatrix},\begin{bmatrix}2\\2\\1\end{bmatrix}\right\}$, $\mathcal{C}=\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}3\\4\\5\end{bmatrix}\right\}$, $[\vec{x}]_C=\begin{bmatrix}-4\\10\\11\end{bmatrix}_C$
   b. $\mathcal{B}=\left\{\begin{bmatrix}1\\-2\\2\\-1\end{bmatrix},\begin{bmatrix}5\\1\\2\\1\end{bmatrix},\begin{bmatrix}2\\2\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\3\\1\end{bmatrix}\right\}$, $\mathcal{C}=\left\{\begin{bmatrix}2\\2\\0\\3\end{bmatrix},\begin{bmatrix}1\\1\\-1\\1\end{bmatrix},\begin{bmatrix}2\\0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\4\end{bmatrix}\right\}$, $[\vec{x}]_B=\begin{bmatrix}5\\2\\-3\\10\end{bmatrix}_B$

6. Find the standard matrix transformation of T for each of the following.
   a. $T: R^2 \to R^4$ $T(\vec{e_1})=(3,1,3,1), T(\vec{e_2})=(-5,2,0,0)$ where $\vec{e_1}=(1,0), \vec{e_2}=(0,1)$.

   b. $T: R^2 \to R^2$ is a vertical shear transformation that maps $\vec{e_1}$ to $\vec{e_1}-3\vec{e_2}$, but leaves $\vec{e_2}$ unchanged.

   c. $T: R^2 \to R^2$ rotates points about origin through -3π/2 radians clockwise.

   d. $T: R^2 \to R^2$ first reflects points through the horizontal x_1-axis and then rotates points –π/2 radians.

7. Let $T: P^2 \to P^3$ be the transformation that maps a polynomial p(t) onto the polynomial (t+3)p(t).
   a. Find the image of $p(t) = 3 - 2t + t^2$.

   b. Show that T is a linear transformation.

   c. Find the matrix T relative to the basis $\{1, t, t^2\}$ and $\{1, t, t^2, t^3\}$.

8. For each of the linear transformations below, write the matrix of the linear transformation.
   a. $T: \vec{x} \in R^3 \longmapsto T(\vec{x}) \in R^3$, where T is given by $T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}2x_1-4x_2\\x_1-x_3\\-x_2+3x_3\end{bmatrix}$.
   b. $T: \vec{x} \in R^2 \longmapsto T(\vec{x}) \in R^3$, where T is given by $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}3x_1-2x_2\\x_1+4x_2\\x_2\end{bmatrix}$.
   c. Consider a polynomial in P_2 given by $p(t) = a_0 + a_1 t + a_2 t^2$. Define a linear operator T by $T(p(t)) = (2t^2 - t + 6)p(t)$ in P_4. Find the matrix of the transformation. [Hint: See Example 2.]
   d. Consider a polynomial in P_3 given by $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$. Find the matrix of the linear transformation taking this vector into P_2 defined by the derivative operator $\frac{d}{dt}[p(t)]$. [Hint: See Example 3.]
   e. Consider the function defined as $y(x) = a_1 e^x + a_2 e^{-x} + a_3 e^{5x} + a_4 e^{-7x}$. Write the matrix of the linear transformation defined by the derivative operator $\frac{d}{dx}[y(x)]$.
   f. Consider a function defined as $y(x) = a_1 e^{3x}\cos(2x) + a_2 e^{3x}\sin(2x)$. Write the matrix of the linear transformation defined by the derivative operator $\frac{d}{dx}[y(x)]$.

g. Find linear transformation matrix that transforms a vector in R^2 by rotating it counterclockwise by 225°.

h. Find a linear transformation matrix that transforms a vector in R^3 by rotating it through an angle $2\pi/3$ in the x_2x_3-plane, then scales the x_1, x_2 directions by a factor of 4 and 2 respectively, and then reflects along the line $x_1 = x_3$.

9. For each of the B bases below, represent the vectors in the coordinate system of the C basis.

a. $\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\}, \mathscr{C} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -7 \end{bmatrix} \right\}$

b. $\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}, \mathscr{C} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \right\}$

c. $\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}, \mathscr{C} = \left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix} \right\}$