12/2/2021

Applications Examples (eigenvalues)
Review for Final

Markov Chain example

$$P = \begin{bmatrix} 0.8 & 0.1 & 0.25 \\ 0.05 & 0.75 & 0.25 \\ 0.15 & 0.15 & 0.5 \end{bmatrix}$$

Equilibrium is the steady state, the long-term trend of the system.

$$Pq = q$$
$$Pq - qI = 0$$
$$(P - I)q = 0$$

$$P - I = \begin{bmatrix} 0.8 - 1 & 0.1 & 0.25 \\ 0.05 & 0.75 - 1 & 0.25 \\ 0.15 & 0.15 & 0.5 - 1 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.1 & 0.25 \\ 0.05 & -0.25 & 0.25 \\ 0.15 & 0.15 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{35}{18} \\ 0 & 1 & -\frac{25}{18} \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - \frac{35}{18}x_3 = 0$$
$$x_2 - \frac{25}{18}x_3 = 0$$
$$x_3 = x_3$$

$$x_1 = \frac{35}{18}x_3$$
$$x_2 = \frac{25}{18}x_3$$
$$x_3 = x_3$$

$$x = \begin{bmatrix} 35 \\ 25 \\ 18 \end{bmatrix}$$

q vector must be probabilities, so it must add to 1:
Add: 35+25+18=68; then divide the whole vector by 78.

$$q = \begin{bmatrix} \frac{35}{78} \\ \frac{25}{78} \\ \frac{18}{78} \end{bmatrix} = \begin{bmatrix} \frac{35}{78} \\ \frac{25}{78} \\ \frac{3}{13} \end{bmatrix}$$

Use this process when the problem says solve "algebraically". You can raise P to a large power until all the columns of the matrix are identical when it doesn't specify. The columns of the matrix are then the equilibrium vector.

Discrete Dynamical Systems

$$A = \begin{bmatrix} 0.3 & 0.4 \\ -0.3 & 1.1 \end{bmatrix}$$

A trajectory is essentially the set of points obtained from plotting the results of

$$x_{n+1} = Ax_n$$

This gives an idea of how the system behaves from a single starting point. But we want the overall behavior from any possible starting point. For that we need eigenvalues and eigenvectors.

$$A - \lambda I = \begin{bmatrix} 0.3 - \lambda & 0.4 \\ -0.3 & 1.1 - \lambda \end{bmatrix}$$

$$(0.3 - \lambda)(1.1 - \lambda) + (0.3)(0.4) = 0$$

$$\lambda^2 - 1.4\lambda + 0.33 + 0.12 = \lambda^2 - 1.4\lambda + 0.45 = 0$$

$$(\lambda - 0.9)(\lambda - 0.5) = 0$$
$$\lambda_1 = 0.9, \lambda_2 = 0.5$$

$$\begin{bmatrix} 0.3 - \lambda_1 & 0.4 \\ -0.3 & 1.1 - \lambda_1 \end{bmatrix} = \begin{bmatrix} 0.3 - 0.9 & 0.4 \\ -0.3 & 1.1 - 0.9 \end{bmatrix} = \begin{bmatrix} -0.6 & 0.4 \\ -0.3 & 0.2 \end{bmatrix}$$

$$-0.3x_1 + 0.2x_2 = 0$$

$$x_1 = \frac{2}{3}x_2$$
$$x_2 = x_2$$

$$v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

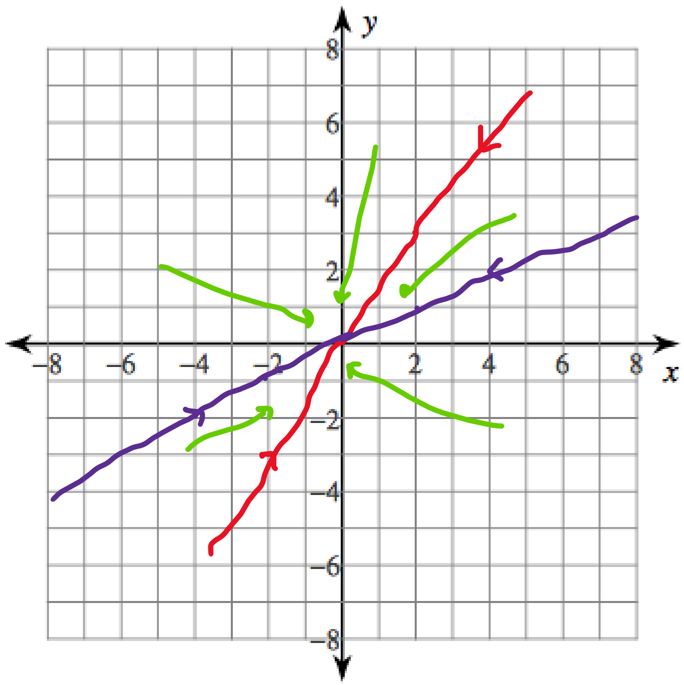
$$\begin{bmatrix} 0.3 - \lambda_2 & 0.4 \\ -0.3 & 1.1 - \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.3 - 0.5 & 0.4 \\ -0.3 & 1.1 - 0.5 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.4 \\ -0.3 & 0.6 \end{bmatrix}$$

$$-0.2x_1 + 0.4x_2 = 0$$

$$x_1 = 2x_2$$
$$x_2 = x_2$$

$$v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

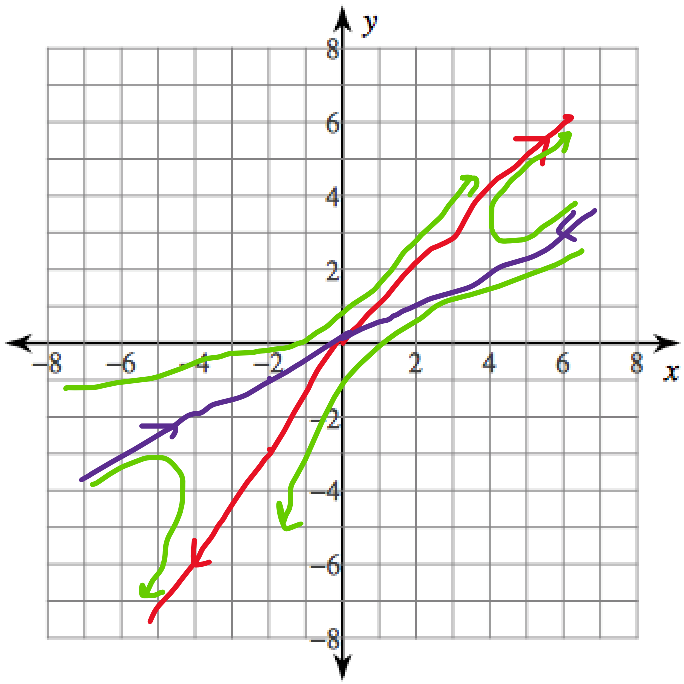
$$x_{n+1} = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \lambda_1^n + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \lambda_2^n$$



this situation with both eigenvalues less than 1 (in absolute value), the origin is an attractor (since all vectors head to zero).

When eigenvalues are greater than 1, the origin is a repeller (all vectors get bigger, arrows point away), so it looks like the reverse of this attractor situation.

If one eigenvalue is bigger than 1, and one is less than one:



Suppose $\lambda_1 = 1.2$, same eigenvector as before, and $\lambda_2 = 0.5$, same eigenvector as before. Here, the origin is a saddle point.

If the eigenvalues are complex, this indicates a rotation. Sample trajectories will spiral around the origin. The magnitude of the eigenvalues: $\lambda = a \pm bi$, $a^2 + b^2 > 1$ then it spirals out, and if $a^2 + b^2 < 1$, the system spiral in.

ODE system

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$

$$Ax = x'$$

Form of the solution $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} e^{\lambda_2 t}$

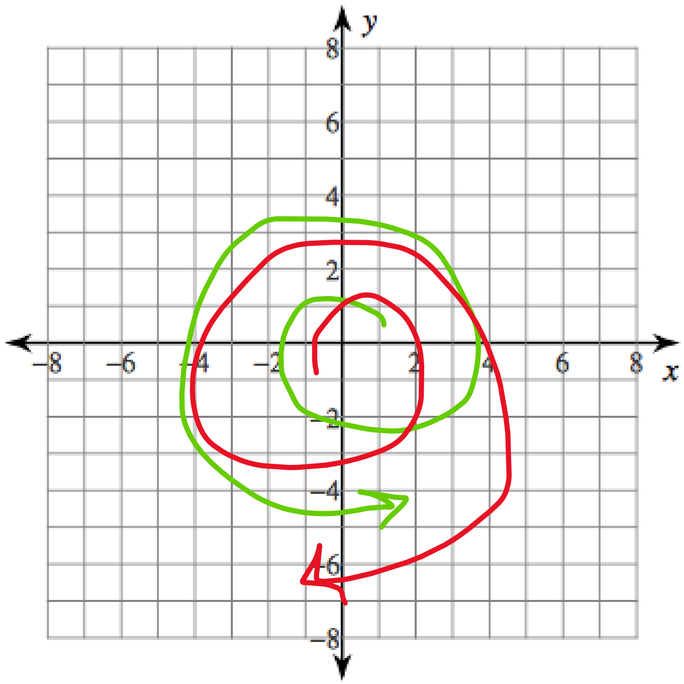
Lambdas are the eigenvalues and the vs are the eigenvectors

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ -2 & 1-\lambda \end{bmatrix}$$

$$(3-\lambda)(1-\lambda) + 2 = 0$$

$$\lambda^2 - 4\lambda + 5 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

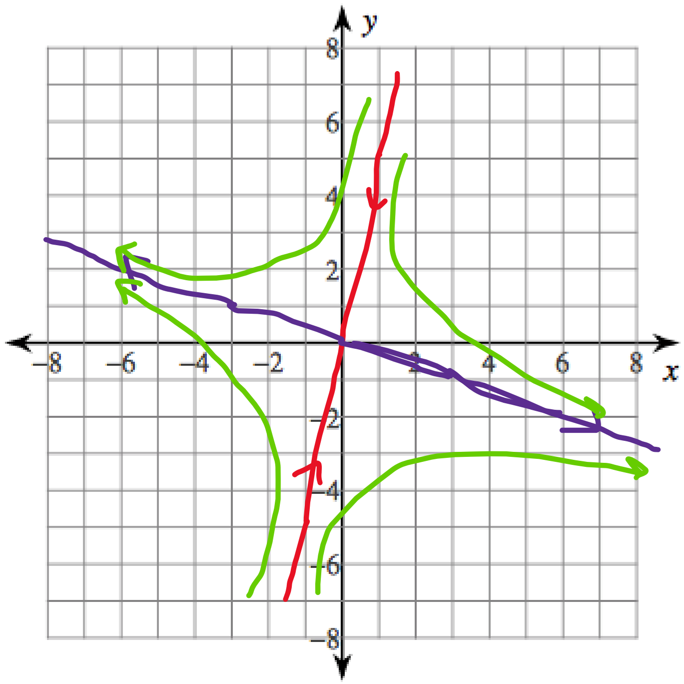


complex eigenvalue is a rotation (don't worry about clockwise or counterclockwise). And the part of the eigenvalue that determine whether the origin attracts or repels is real part. If the real part is positive, then it spiral outward. If the real part is negative, the spiral is inward.

When the solution is complex, the complex/imaginary part become a sine and cosine, and the real part is the e exponent.

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} e^{at} \cos(bt) + c_2 \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} e^{at} \sin(bt)$$

If the eigenvalues are both real, then the procedure is mostly similar to the discrete case. Plot eigenvectors. If the eigenvalue is real and negative, the origin is an attractor. If the eigenvalue is real and positive, the origin will be a repeller. If they are different signs, then the origin is a saddle point.



Suppose $\lambda_1 = -2$, and $\lambda_2 = \frac{3}{2}$, and $v_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$