08/30/2022

Arc Length and Surface Area (2.4)
Applications?

Arc length is the length of the curve, along the curve and not the straight-line distance between starting and stopping.

The detailed derivation is in the Arc Length handout.

The formula for the arc length is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

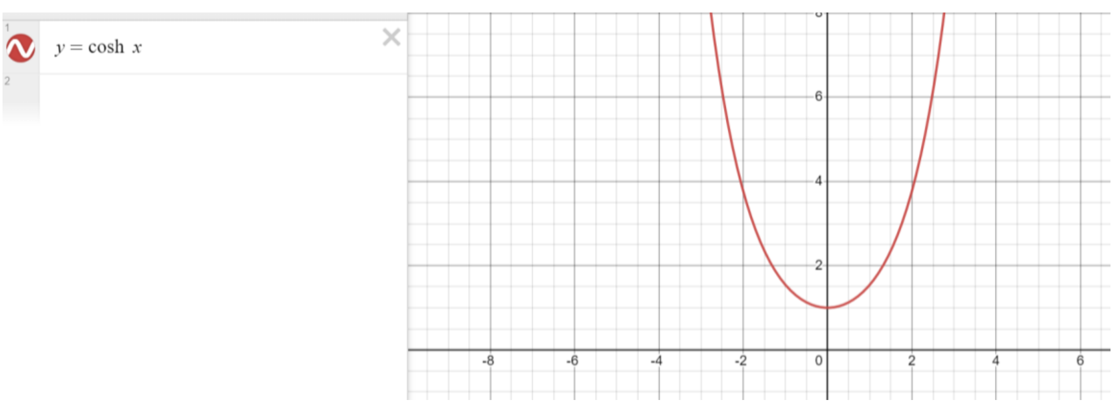
Example. Use the arc length formula to find the length of the curve $y = 3x + 1$ between x=1 and x=2.

$$y = f(x) = 3x + 1$$
$$f'(x) = 3$$

$$s = \int_1^2 \sqrt{1 + 3^2} dx = s = \int_1^2 \sqrt{10} dx = \sqrt{10}x\Big|_1^2 = 2\sqrt{10} - \sqrt{10} = \sqrt{10}$$

$$(1, f(1) = 4), (2, f(2) = 7)$$
$$d = \sqrt{(2-1)^2 + (7-4)^2} = \sqrt{1+9} = \sqrt{10}$$

Example. Find the length of the curve on $f(x) = \cosh x$ between x=0 and x=2.



$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\cos^2 x + \sin^2 x = 1$$
$$\cosh^2 x - \sinh^2 x = 1$$

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

$$f'(x) = \sinh x$$

$$s = \int_a^b \sqrt{1 + [\sinh x]^2} dx = \int_0^2 \sqrt{1 + \sinh^2 x}\, dx = s = \int_0^2 \sqrt{\cosh^2 x}\, dx = \int_0^2 \cosh x\, dx = \sinh x|_0^2 =$$

$$\left.\frac{e^x - e^{-x}}{2}\right|_0^2 = \frac{1}{2}(e^2 - e^{-2} - 1 + 1) = \frac{e^2 - \frac{1}{e^2}}{2}$$

Example. $f(x) = \ln\ |\sec x\,|$, or $g(x) = \ln\ |\cos x\,|$

$$f'(x) = \frac{1}{\sec x} \times \sec x \tan x = \tan x$$

$$g'(x) = \frac{1}{\cos x} \times -\sin x = -\tan x$$

$$s = \int_a^b \sqrt{1 + [\tan x]^2} dx = \int_a^b \sqrt{1 + [-\tan x]^2} dx = \int_a^b \sqrt{1 + \tan^2 x}\, dx = \int_a^b \sqrt{\sec^2 x}\, dx = \int_a^b \sec x\, dx$$

$$= [\ln|\sec x + \tan x|]_a^b$$

One other class of functions that you can also do by hand in the arc length formula.

$$f(x) = \frac{x^3}{3} + \frac{1}{4x} = \frac{1}{3}x^3 + \frac{1}{4}x^{-1}$$

$$f'(x) = x^2 - \frac{1}{4}x^{-2}$$

$$1 + \left(x^2 - \frac{1}{4}x^{-2}\right)^2 = 1 + \left(x^2 - \frac{1}{4}x^{-2}\right)\left(x^2 - \frac{1}{4}x^{-2}\right) =$$

$$1 + x^4 - \frac{1}{4} - \frac{1}{4} + \frac{1}{16}x^{-4} = 1 + \left(x^4 - \frac{1}{2} + \frac{1}{16}x^{-4}\right) = x^4 + \frac{1}{2} + \frac{1}{16}x^{-4} = \left(x^2 + \frac{1}{4}x^{-2}\right)^2$$

After the radical and the square cancel in the arc length formula, this is two power terms that can be easily integrated.

Example. Common functions are not integrable.

$$f(x) = x^2 + 3x$$
$$f'(x) = 2x + 3$$

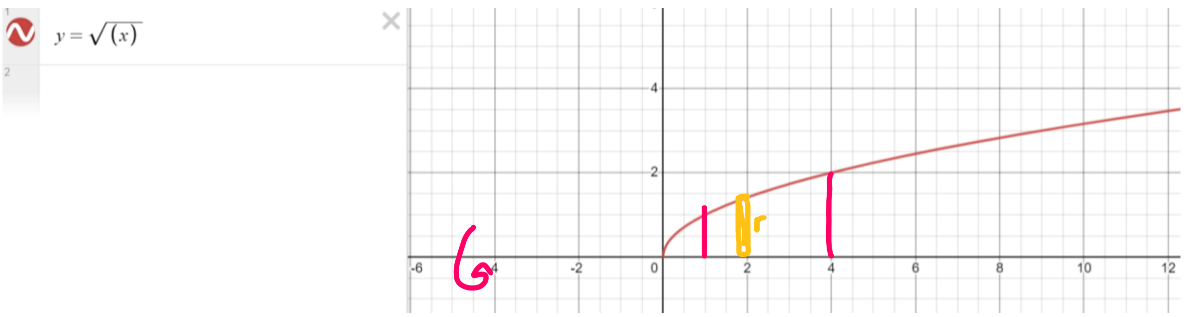
$$\sqrt{1 + (2x + 3)^2} = \sqrt{4x^2 + 6x + 10}$$

In most cases, we have to resort to numerical integration techniques.

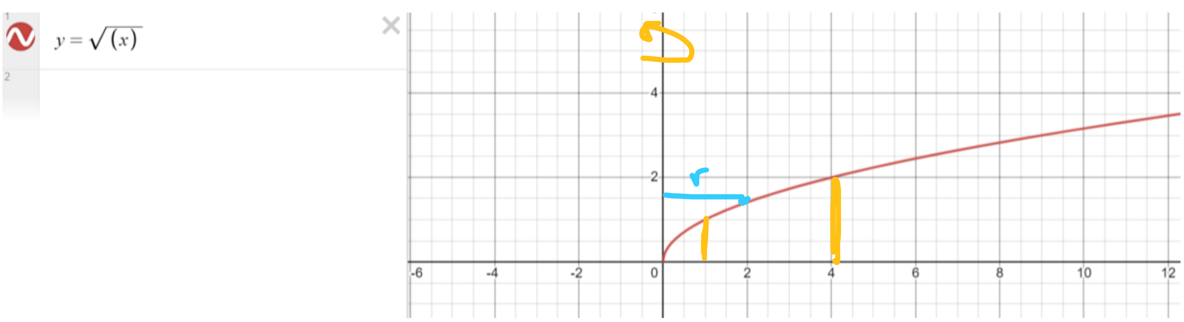
Surface area of solids of revolution.

Example. $f(x) = \sqrt{x}$, rotated around the x-axis between x=1 and x=4. Find the surface area of the solid of revolution obtained.

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}dx$$



In this configuration, the radius is the height of the original function. $r(x) = f(x)$



In this configuration, the radius is the variable, x (the distance from the point on the curve to the y-axis). $r(x) = x$.

For this problem:
$f(x) = \sqrt{x}$, rotated around the x-axis between x=1 and x=4. Find the surface area of the solid of revolution obtained.

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}dx = S = 2\pi \int_1^4 \sqrt{x}\sqrt{1 + \left[\frac{1}{2\sqrt{x}}\right]^2}\,dx = 2\pi \int_1^4 \sqrt{x}\sqrt{1 + \frac{1}{4x}}dx =$$

$$2\pi \int_1^4 \sqrt{1x + \frac{x}{4x}}dx = 2\pi \int_1^4 \sqrt{x + \frac{1}{4}}dx = 2\pi\left(\frac{2}{3}\right)\left(x + \frac{1}{4}\right)^{\frac{3}{2}}\Bigg|_1^4 = \frac{4\pi}{3}\left[\left(\frac{17}{4}\right)^{\frac{3}{2}} - \left(\frac{5}{4}\right)^{\frac{3}{2}}\right]$$

Very ugly answers.

If we changed our example to:
$f(x) = \sqrt{x}$, rotated around the y-axis between x=1 and x=4. Find the surface area of the solid of revolution obtained.

$$S = 2\pi \int_1^4 x \sqrt{1 + \left[\frac{1}{2\sqrt{x}}\right]^2} dx = 2\pi \int_1^4 x \sqrt{1 + \frac{1}{4x}} dx$$

Requires integration by parts is needed. Alternatively, you can do a change of variables.

$$u = \sqrt{1 + \frac{1}{4x}}$$

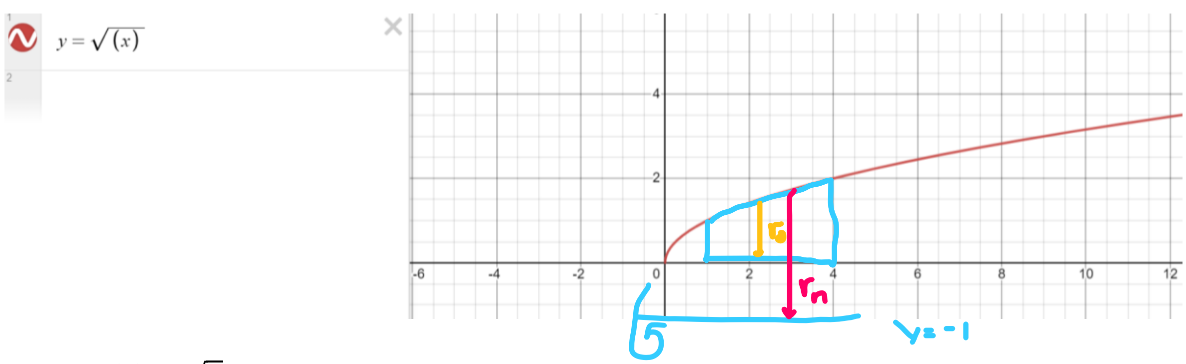
$$u^2 = 1 + \frac{1}{4x} \to u^2 - 1 = \frac{1}{4x} \to 4u^2 - 4 = \frac{1}{x} \to x = \frac{1}{4u^2 - 4}$$

$$dx = \frac{1}{(4u^2 - 4)^2}(-8u)du$$

Replace these in the integrable and perhaps could integrate from there.

If the functions are in terms of y-variables, then r(y) is the function if you are rotating around the y-axis, then r(y)=f(y), and and r(y)=y if you are rotating around the x-axis.

Similarly, if you are rotating around a non-zero axis (not the x-axis and not the y-axis), then choose the axis parallel to your axis of rotation and set it up that way. Take into account the r(x) minus the axis of rotation.

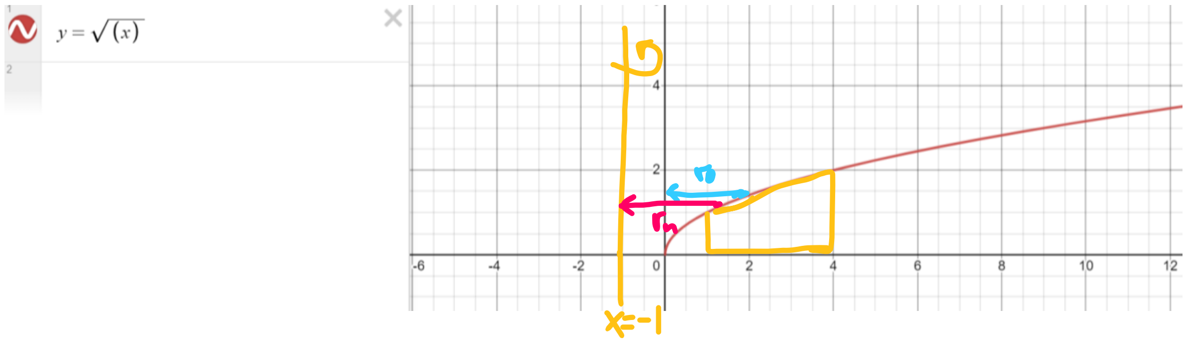


Rotate $f(x) = \sqrt{x}$ around y=-1

Nothing about this changes the arclength portion of the expression. All that changes is the radius.

$$r(x) = f(x) - (-1)$$

Function minus the axis of rotation.



Rotate $f(x) = \sqrt{x}$ around x=-1

The new radius needs to take into account the extra distance to the new radius.

$$r(x) = x - (-1)$$

Variable minus the axis of rotation.

Switch the order of subtraction if the rotation axis is on the opposite side of the region.

Probability applications.

Probability density functions: continuous f(x) and the area under the curve has a total area of 1.

Given a function $f(x) = kx^2, 0 \leq x \leq 4$, (the function is assumed to be zero everywhere else).

To make this a probability density function, we need to determine the value of k that makes the area equal to 1.

$$\int_0^4 kx^2 dx = \frac{k}{3}x^3\Big|_0^4 = \frac{k}{3}[64] = 1 \rightarrow k = \frac{3}{64}$$

$$f(x) = \frac{3}{64}x^2, 0 \leq x \leq 4$$

$$P(x > 3) = \int_3^4 \frac{3}{64}x^2 dx = \frac{3}{64}\left[\frac{1}{3}x^3\right]_3^4 = \frac{1}{64}[64 - 27] = \frac{37}{64} \approx 0.578$$

Normal distribution
Density function (standard normal)

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

We'll circle back to this to calculate the mean of probability density functions.