9/13/2022

Finish Trig Integrals (secant/tangent)
Trig Substitution (3.3)

Powers of secants and tangents:

1) Even powers of secant.
   Pull out one $\sec^2 x$, then replace the even secants that remain with tangents using the Pythagorean identity

$$1+\tan^2 x=\sec^2 x$$

   Example.

$$\int \sec^4 x\,dx$$

$$\int \sec^2 x\,(\sec^2 x)dx=\int (1+\tan^2 x)\sec^2 x\,dx$$

$$u=\tan x\,,du=\sec^2 x\,dx$$

$$\int 1+u^2du=u+\frac{1}{3}u^3+C=\tan x+\frac{1}{3}\tan^3 x+C$$

2) Odd power of tangent in combination with secants.
   Pull out a secxtanx, and replace the remaining even powers of tangent with secants.

   Example.

$$\int \sec^4 x\tan^3 x$$

$$\int \sec x\tan x\,(\sec^3 x)(\tan^2 x)dx=\int \sec^3 x\,(\sec^2 x-1)(\sec x\tan x)dx$$

$$u=\sec x\,,du=\sec x\tan x\,dx$$

$$\int u^3(u^2-1)du=\int u^5-u^3du=\frac{1}{6}u^6-\frac{1}{4}u^4+C=\frac{1}{6}\sec^6 x-\frac{1}{4}\sec^4 x+C$$

3) Odd powers of secant (possibly with even powers of tangent)
   Convert everything to secants and then for powers greater than 1, use integration by parts.

$$\int \sec x\,dx=\ln|\sec x+\tan x|+C$$

   Example.

$$\int \sec^3 dx=\int \sec x\,(\sec^2 x)dx$$

$$u=\sec x\,,dv=\sec^2 x\,dx$$
$$du=\sec x\tan x\,dx,v=\tan x$$

$$\sec x \tan x - \int \tan x\,(\sec x \tan x) dx =$$
$$\sec x \tan x - \int \sec x \tan^2 x\, dx$$

$$\sec x \tan x - \int \sec x\,(\sec^2 x - 1) dx = \sec x \tan x - \int \sec^3 x - \sec x\, dx$$

$$\int \sec^3 dx = \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx$$

$$\int \sec^3 dx = \sec x \tan x + \ln\ |\sec x + \tan x| - \int \sec^3 x\, dx$$

$$2\int \sec^3 x\, dx = \sec x \tan x + \ln\ |\sec x + \tan x|$$

$$\int \sec^3 x\, dx = \frac{1}{2}[\sec x \tan x + \ln|\sec x + \tan x|] + C$$

If you don't know what to do, convert everything to sine and cosine, especially if you have three or more different trig functions.

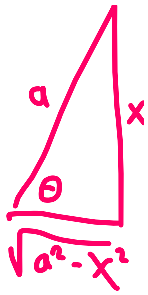
Trig Substitution
Generally, this technique is used to integrate functions containing square roots with sums or differences of squares under the root.

If you see $\sqrt{a^2 - x^2}$ use $x = a \sin \theta$
If you see $\sqrt{x^2 - a^2}$ use $x = a \sec \theta$
If you see $\sqrt{a^2 + x^2}$ use $x = a \tan \theta$

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1 - x^2}}$$
$$\frac{d}{dx}[\operatorname{arcsec} x] = \frac{1}{|x|\sqrt{x^2 - 1}}$$
$$\frac{d}{dx}[\arctan x] = \frac{1}{1 + x^2}$$

$$\sqrt{a^2 - x^2}$$
$$x = a \sin \theta$$
$$\sqrt{a^2 - a^2 \sin^2 \theta} = \sqrt{a^2(1 - \sin^2 \theta)} = \sqrt{a^2 \cos^2 \theta} = a \cos \theta$$

$$\frac{x}{a} = \sin \theta$$




$$\sqrt{x^2 - a^2}$$
$$x = a\sec\theta$$
$$\sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = a\tan\theta$$

Likewise, for $\sqrt{a^2 + x^2}$, with $x = a\tan\theta$

$$\sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(1 + \tan^2\theta)} = \sqrt{a^2\sec^2\theta} = a\sec\theta$$

Example.

$$\int \sqrt{9 - x^2}dx$$

$$a = 3$$
$$x = 3\sin\theta$$
$$\sqrt{9 - x^2} = 3\cos\theta$$
$$dx = 3\cos\theta\, d\theta$$

$$\int \sqrt{9 - x^2}dx = \int 3\cos\theta\,(3\cos\theta)d\theta = 9\int \cos^2\theta\, d\theta = 9\left(\frac{1}{2}\right)\int 1 + \cos 2\theta\, d\theta =$$

$$\frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

$$x = 3\sin\theta$$
$$\frac{x}{3} = \sin\theta$$
$$\arcsin\left(\frac{x}{3}\right) = \theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\frac{9}{2}\left[\arcsin\left(\frac{x}{3}\right) + \frac{1}{2}(2\sin\theta\cos\theta)\right] + C = \frac{9}{2}\left[\arcsin\left(\frac{x}{3}\right) + \sin\theta\cos\theta\right] + C =$$
$$\frac{9}{2}\left[\arcsin\left(\frac{x}{3}\right) + \left(\frac{x}{3}\right)\left(\frac{\sqrt{9 - x^2}}{3}\right)\right] + C = \frac{9}{2}\left[\arcsin\left(\frac{x}{3}\right) + \frac{x\sqrt{9 - x^2}}{9}\right] + C =$$

$$\frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{x\sqrt{9 - x^2}}{2} + C$$

Do not leave thetas behind.

Do not leave trig functions with other inverse trig functions inside them. Use the triangle to simplify.

Example.

$$\int \frac{\sqrt{3+4x^2}}{2x}\,dx$$

$$u = 2x, du = 2dx, \frac{1}{2}du = dx$$

$$\frac{1}{2}\int \frac{\sqrt{3+u^2}}{u}\,du$$

$$u = \sqrt{3}\tan\theta \to \frac{u}{\sqrt{3}} = \tan\theta = \frac{2x}{\sqrt{3}} = \tan\theta$$

$$du = \sqrt{3}\sec^2\theta\,d\theta$$

$$\sqrt{3+u^2} = \sqrt{3}\sec\theta \to \frac{\sqrt{3+u^2}}{\sqrt{3}} = \sec\theta = \frac{\sqrt{3+4x^2}}{\sqrt{3}}$$

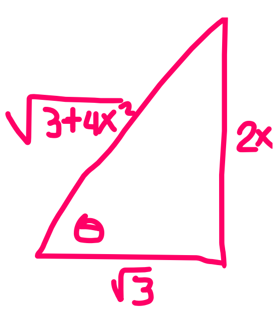
$$\frac{1}{2}\int \frac{\sqrt{3}\sec\theta}{\sqrt{3}\tan\theta}\sqrt{3}\sec^2\theta\,d\theta = \frac{\sqrt{3}}{2}\int \frac{\sec\theta\sec^2\theta}{\tan\theta}\,d\theta = \frac{\sqrt{3}}{2}\int \frac{\sec\theta\,(1+\tan^2\theta)}{\tan\theta}\,d\theta =$$

$$\frac{\sqrt{3}}{2}\int \frac{\sec\theta + \sec\theta\tan^2\theta}{\tan\theta}\,d\theta = \frac{\sqrt{3}}{2}\int \frac{\sec\theta}{\tan\theta} + \frac{\sec\theta\tan^2\theta}{\tan\theta}\,d\theta = \frac{\sqrt{3}}{2}\int \frac{\sec\theta}{\tan\theta} + \sec\theta\tan\theta\,d\theta$$

$$\frac{\sqrt{3}}{2}\int \sec\theta\cot\theta + \sec\theta\tan\theta\,d\theta = \frac{\sqrt{3}}{2}\int \frac{1}{\cos\theta}\left(\frac{\cos\theta}{\sin\theta}\right) + \sec\theta\tan\theta\,d\theta$$

$$\frac{\sqrt{3}}{2}\int \frac{1}{\sin\theta} + \sec\theta\tan\theta\,d\theta = \frac{\sqrt{3}}{2}\int \csc\theta + \sec\theta\tan\theta\,d\theta =$$

$$\frac{\sqrt{3}}{2}[\ln|\csc\theta - \cot\theta| + \sec\theta] + C =$$




$$\frac{\sqrt{3}}{2}\left[\ln\left|\frac{\sqrt{3+4x^2}}{2x} - \frac{\sqrt{3}}{2x}\right| + \frac{\sqrt{3+4x^2}}{\sqrt{3}}\right] + C$$

$$\frac{\sqrt{3}}{2}\left[\ln\left|\frac{\sqrt{3+4x^2}-\sqrt{3}}{2x}\right| + \frac{\sqrt{3+4x^2}}{\sqrt{3}}\right] + C$$

$$\frac{\sqrt{3}}{2}\left[\ln\left|\sqrt{3+4x^2}-\sqrt{3}\right|-\ln\ |2x|+\frac{\sqrt{3+4x^2}}{\sqrt{3}}\right]+C$$

Example.

$$\int \frac{1}{\sqrt{x^2-4}}\,dx$$

$$x = 2\sec\theta$$
$$\sqrt{x^2-4} = 2\tan\theta$$
$$dx = 2\sec\theta\tan\theta\,d\theta$$

$$\int \frac{1}{\sqrt{x^2-4}}\,dx = \int \frac{2\sec\theta\tan\theta}{2\tan\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta|+C$$

$$= \ln\left|\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}\right|+C$$

These techniques do not strictly require radicals.

$$(1+x^2) = \left(\sqrt{1+x^2}\right)^2$$
$$(1+x^2)^{\frac{3}{2}} = \left(\sqrt{1+x^2}\right)^3$$

$$(1-x^2)^{\frac{5}{2}} = \left(\sqrt{1-x^2}\right)^5$$

See handout for more worked examples.

Next time Partial Fractions.