9/8/2022

Integration by Parts (3.1)
Trig Integrals (3.2)

Integration by parts is essentially reverse engineering the product rule, but backwards.

$$(uv)' = u'v + uv'$$

$$\int u dv = \int u(x)v'(x)dx$$

Think of $u = u(x), dv = v'(x)dx$

Integrate both sides of the product rule:

$$uv = \int (uv)' = \int u'v + uv' = \int u'v + \int uv'$$

$$uv = \int u'v + \int uv'$$

$$\int uv' = uv - \int u'v$$

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$

Example.

$$\int xe^x dx$$

$$u = x, dv = e^x dx$$
$$du = dx, v = \int dv = \int e^x dx = e^x$$

$$\int u dv = uv - \int v du$$

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

Does this really work?

$$(xe^x - e^x + C)' =? \, xe^x$$
$$e^x + xe^x - e^x + 0 = xe^x$$

Rule of Thumb for choosing u and dv

LIATE

Logarithms, Inverse Trig, Algebraic, Trigonometric, Exponential
u ⟷ dv

Algebraic can be split into subcategories: Polynomial terms, Rational or Root functions

Example.

$$\int \ln(x)\,dx$$

$$u = \ln(x),\, dv = dx$$
$$du = \frac{1}{x}dx, v = x$$

$$\int u dv = uv - \int v du$$

$$\int \ln(x)\,dx = x\ln(x) - \int x\frac{1}{x}dx$$

$$= x\ln(x) - \int 1 dx = x\ln(x) - x + C$$

Example.

$$\int \arcsin(x)\,dx$$

$$u = \arcsin(x),\, dv = dx$$
$$du = \frac{1}{\sqrt{1-x^2}}dx, v = x$$

$$\int \arcsin(x)\,dx = x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}dx$$

$$w = 1 - x^2, dw = -2xdx, -\frac{1}{2}dw = xdx$$

$$x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}dx = x\arcsin(x) - \int -\frac{1}{2}\left(w^{-\frac{1}{2}}\right)dw = x\arcsin x + w^{\frac{1}{2}} + C$$

$$= x\arcsin x + \sqrt{1-x^2} + C$$

Example.

$$\int x^3 e^{x^2} dx = \int x^2\left(xe^{x^2}\right)dx$$

$$u = x^2, dv = xe^{x^2}dx$$

$$du = 2xdx, v = \int xe^{x^2} dx = \frac{1}{2}e^{x^2}$$

$$w = x^2, dw = 2xdx, \frac{1}{2}dw = xdx$$

$$\int xdx\, e^{x^2} = \int \frac{1}{2}dw\, e^w = \frac{1}{2}\int e^w dw = \frac{1}{2}e^w = \frac{1}{2}e^{x^2}$$

$$\int x^3 e^{x^2} dx = \frac{1}{2}x^2 e^{x^2} - \int \frac{1}{2}e^{x^2}(2xdx) = \frac{1}{2}x^2 e^{x^2} - \int xe^{x^2} dx = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C$$

Example.

$$\int e^x \cos x\, dx$$

$$u = \cos x\,, dv = e^x dx$$
$$du = -\sin x\, dx, v = e^x$$

$$\int e^x \cos x\, dx = e^x \cos x - \int -e^x \sin(x)\, dx$$

$$u = \sin(x)\,, dv = e^x dx$$
$$du = \cos x\, dx, v = e^x$$

$$\int e^x \cos x\, dx = e^x \cos x + e^x \sin(x) - \int e^x \cos(x)\, dx$$

The trick here is now to add the two integrals that are identical to each other on the same side of the equation.

$$\int e^x \cos x\, dx + \int e^x \cos(x)\, dx = e^x \cos x + e^x \sin(x) - \int e^x \cos(x)\, dx + \int e^x \cos(x)\, dx$$

$$2\int e^x \cos(x)\, dx = e^x \cos x + e^x \sin(x)$$

$$\int e^x \cos(x)\, dx = \frac{1}{2}[e^x \cos x + e^x \sin(x)] + C$$

Secant-cubed also behaves like this.

Example.

$$\int x^3 \sin(x)\, dx$$

Long way.

$$u = x^3, dv = \sin(x)\, dx$$
$$du = 3x^2 dx, v = -\cos(x)$$

$$-x^3\cos(x) - \int -3x^2\cos(x)\,dx$$

$$u = 3x^2, dv = \cos(x)\,dx$$
$$du = 6x dx, v = \sin(x)$$

$$-x^3\cos(x) + \int 3x^2\cos(x)\,dx = -x^3\cos(x) + 3x^2\sin(x) - \int 6x\sin(x)\,dx$$

$$u = 6x, dv = \sin(x)\,dx$$
$$du = 6dx, v = -\cos(x)$$

$$-x^3\cos(x) + 3x^2\sin(x) - \left[-6x\cos(x) - \int -6\cos(x)\,dx\right] =$$
$$-x^3\cos(x) + 3x^2\sin(x) + 6x\cos(x) - \int 6\cos(x)\,dx$$
$$-x^3\cos(x) + 3x^2\sin(x) + 6x\cos(x) - 6\sin(x) + C$$

Short way: using the tabular method.

| +/- | $u$ | $dv$ |
|---|---|---|
| + | $x^3$ | $\sin(x)$ |
| - | $3x^2$ | $-\cos(x)$ |
| + | $6x$ | $-sin(x)$ |
| - | $6$ | $\cos(x)$ |
| + | $0$ | $\sin(x)$ |

$$-x^3\cos(x) + 3x^2\sin(x) + 6x\cos(x) - 6\sin(x) + C$$

Use with polynomials for u, and dv's that can be integrated without borrowing or canceling.

Trig Integrals

Cosines and Sines

1) All powers of sine and cosine are even.
   Use power reducing identities until you eliminate all the square powers (only linear powers left).

$$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$$

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

$$\int \cos^2 x \sin^2 x\,dx = \int \frac{1}{2}(1 + \cos(2x))\frac{1}{2}(1 - \cos(2x))dx = \frac{1}{4}\int 1 - \cos^2 2x\,dx$$

$$= \frac{1}{4}\int 1 - \left[\frac{1}{2}(1 + \cos(4x))\right]dx = \frac{1}{4}\int \frac{1}{2} - \frac{1}{2}\cos 4x\,dx = \frac{1}{4}\left[\frac{1}{2}x - \frac{1}{8}\sin(4x)\right] + C$$

2) If one of the powers is odd, then pull out one odd power (1), and replace the remaining terms using the Pythagorean identity.

$$\sin^2 x + \cos^2 x = 1$$

$$\int \sin^3 x \cos^5 x \, dx = \int \sin(x) \, [\sin^2 x] \cos^5 x \, dx = \int \sin(x) \, [1 - \cos^2 x] \cos^5 x \, dx$$

$$u = \cos(x) \, , du = -\sin(x) \, dx$$

$$-\int (1 - u^2)(u^5) du = \int -u^5 + u^7 du = -\frac{1}{6}u^6 + \frac{1}{8}u^8 + C = \frac{1}{8}\cos^8 x - \frac{1}{6}\cos^6 x + C$$

If you have 3 or more different trig functions, use identities to make everything sine and cosine before proceeding

Or if have trig functions that don't relate directly through a Pythagorean identity, then converting to sines and cosines is a good strategy

We will continue next time with secants and tangents (cotangent and cosecant will have the same rules).