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Laplace Transforms continued
Unit Step Function
Convolutions

Piecewise function grapher: https://www.graphfree.com/

The Laplace Transform of a piecewise function

$$L(f) = \int_0^\infty e^{-st} f(t)dt$$

Piecewise function:

$$f(t) = \begin{cases} 2t+1, & 0 \le t < 2 \\ 3t, & t \ge 2 \end{cases}$$

$$f(t) = \begin{cases} 0, & t < 0 \\ 2t+1, & 0 \le t < 2 \\ 3t, & t \ge 2 \end{cases}$$

If your piecewise function contains a domain that extends below zero, you can ignore it. Laplace transforms by definition start at 0 or above.

$$L(f) = \int_0^2 e^{-st}(2t+1)dt + \int_2^\infty e^{-st}(3t)dt =$$

$$u = 2t+1, dv = e^{-st}dt \qquad u = 3t, dv = e^{-st}dt$$

$$du = 2dt, v = -\frac{1}{s}e^{-st} \qquad u = 3dt, v = -\frac{1}{s}e^{-st}$$

$$-\frac{1}{s}e^{-st}(2t+1) + \int_0^2 \frac{1}{s}e^{-st}(2)dt - \frac{1}{s}e^{-st}(3t) + \int_2^\infty \frac{1}{s}e^{-st}(3)dt =$$

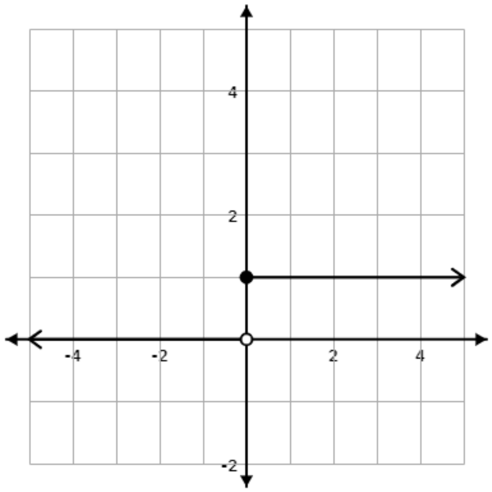
$$-\frac{1}{s}e^{-st}(2t+1) - \frac{2}{s^2}e^{-st}\Big|_0^2 - \frac{1}{s}e^{-st}(3t) - \frac{3}{s^2}e^{-st}\Big|_2^\infty =$$

$$-\frac{1}{s}e^{-2s}(5) - \frac{2}{s^2}e^{-2s} + \frac{1}{s}(1)(1) + \frac{2}{s^2}(1) - (0) - (0) + \frac{1}{s}e^{-2s}(6) + \frac{3}{s^2}e^{-2s}$$

$$e^{-2s}\left(-\frac{5}{s} + \frac{6}{s}\right) + e^{-2s}\left(-\frac{2}{s^2} + \frac{3}{s^2}\right) + \frac{1}{s} + \frac{2}{s^2} = \frac{1}{s} + \frac{2}{s^2} + e^{-2s}\left(\frac{1}{s} + \frac{1}{s^2}\right)$$

To use the Laplace Transform table, we need be able to write the piecewise functions in terms of the unit step function (or Heaviside function).

$$H(t) = u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



$$H(t - c) = u(t - c) = u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

c is any positive value

Transform other piecewise functions into unit step notation, we first have to make the start of the function equal to 0.

$$f(t) = \begin{cases} 2t + 1, & 0 \leq t < 2 \\ 3t, & t \geq 2 \end{cases}$$

In this case, subtract $2t + 1$ from both pieces and pull that value out front.

$$f(t) = (2t + 1) + \begin{cases} 2t + 1 - (2t + 1), & 0 \leq t < 2 \\ 3t - (2t + 1), & t \geq 2 \end{cases}$$

$$f(t) = (2t + 1) + \begin{cases} 0, & 0 \leq t < 2 \\ t - 1, & t \geq 2 \end{cases}$$

Second, "factor" out the expression in the second piece and make it a multiplier of the piecewise part.

$$f(t) = (2t + 1) + (t - 1)\begin{cases} 0, & 0 \leq t < 2 \\ 1, & t \geq 2 \end{cases}$$

Replace piecewise function with unit step notation.

$$f(t) = (2t + 1) + (t - 1)u(t - 2)$$

$$f(t) = (2t + 1) + (t - 1)u_2(t)$$

If there are more than two pieces, you will have to repeat this process for successive pieces (each time there is a jump, you will need a new unit step function.

Example.

$$f(t) = \begin{cases} 1, & 0 \le t < 2 \\ -2t+1, & 2 \le t < 3 \\ 3t, & 3 \le t < 5 \\ t-1, & t \ge 5 \end{cases}$$

Make the first piece 0 by subtracting off whatever the value is.

$$f(t) = 1 + \begin{cases} 0, & 0 \le t < 2 \\ -2t, & 2 \le t < 3 \\ 3t-1, & 3 \le t < 5 \\ t-2, & t \ge 5 \end{cases}$$

In terms of the unit step function for the first piece:

$$f(t) = 1 - 2t\, u(t-2) + function\, u(t-3) + function\, u(t-5)$$

The functions multiplying successive steps are the difference from the previous step to the new step. And then factor that function so that the piece in the stepwise is 1.

Next function: $\left(3t - 1 - (-2t)\right) = stop\ function\ minus\ start\ function$
=$5t - 1$

$$f(t) = 1 - 2t\, u(t-2) + (5t-1)\, u(t-3) + function\, u(t-5)$$

Next function: $\left((t-2) - (3t-1)\right) = t - 2 - 3t + 1 = -2t - 1$

$$f(t) = 1 - 2t\, u(t-2) + (5t-1)\, u(t-3) + (-2t-1)\, u(t-5)$$

$$f(t) = 1 + (-2t)\begin{cases} 0, 0 \le t < 2 \\ 1, \quad t \ge 2 \end{cases} + (5t-1)\begin{cases} 0, & 0 \le t < 3 \\ 1, & t \ge 3 \end{cases} + (-2t-1)\begin{cases} 0, 0 \le t < 5 \\ 1, \quad t \ge 5 \end{cases}$$

$$f(t) = \begin{cases} 1, \quad 0 \le t < 2 \\ 1-2t,\ \ t \le 2 < 3 \\ 1 - 2t + 5t - 1 = 3t, \quad 3 \le t < 5 \\ 1 - 2t + 5t - 1 + (-2t) - 1 = t - 1, t \le 5 \end{cases}$$

Which reconstructs the effects of the original piecewise function.

Now that we can decompose any piecewise function in terms of the unit step function, we can now apply the Laplace transform table to avoid having to use the definition to find the Laplace transform of any piecewise function.

$\boldsymbol{u_c(t) = u(t-c)}$	$\frac{e^{-cs}}{s}$
$\boldsymbol{\delta(t)}$	1
$\boldsymbol{\delta(t-c)}$	e^{-cs}
$\boldsymbol{\delta^{(n)}}$	s^n
$\boldsymbol{u_c(t)f(t-c)}$	$e^{-cs}F(s)$
$\boldsymbol{u_c(t)g(t)}$	$e^{-cs}\mathfrak{L}\{g(t+c)\}$

A section of the Laplace transform table for unit step functions.

Looking at the formula for the general unit step function, if $c = 0$, the transformation is $\frac{1}{s}$ which is the same as the transformation for $f(t) = 1$.

If the jump happens when $c > 0$, then the e^{-cs} term pops up.

Let's go back to the first example that we did by the definition.

$$f(t) = \begin{cases} 2t+1, & 0 \leq t < 2 \\ 3t, & t \geq 2 \end{cases}$$

In terms of the unit step function we found this:

$$f(t) = (2t+1) + (t-1)u_2(t)$$

$$F(s) = 2\left(\frac{1}{s^2}\right) + \frac{1}{s} + e^{-2s}L\big((t+2)-1\big) = 2\left(\frac{1}{s^2}\right) + \frac{1}{s} + e^{-2s}L(t+1) = 2\left(\frac{1}{s^2}\right) + \frac{1}{s} + e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

From integration:

$$\frac{1}{s} + \frac{2}{s^2} + e^{-2s}\left(\frac{1}{s} + \frac{1}{s^2}\right)$$

Example.

$$f(t) = \begin{cases} 1, & 0 \leq t < 2 \\ -2t+1, & 2 \leq t < 3 \\ 3t, & 3 \leq t < 5 \\ t-1, & t \geq 5 \end{cases}$$

Find the Laplace transform using the table.

The function in terms of the unit step function

$$f(t) = 1 - 2t\,u(t-2) + (5t-1)\,u(t-3) + (-2t-1)\,u(t-5)$$

$$f(t) = 1 + (-2t)\,u_2(t) + (5t-1)u_3(t) + (-2t-1)u_5(t)$$

$u_c(t)g(t)$	$e^{-cs}\mathfrak{L}\{g(t+c)\}$

$$F(s) = \frac{1}{s} + e^{-2s}L\big(-2(t+2)\big) + e^{-3s}L(5(t+3)-1) + e^{-5s}L(-2(t+5)-1) =$$

$$\frac{1}{s} + e^{-2s}L(-2t-4) + e^{-3s}L(5t-14) + e^{-5s}L(-2t-11) =$$

$$\frac{1}{s} + e^{-2s}\left(-\frac{2}{s^2} - \frac{4}{s}\right) + e^{-3s}\left(\frac{5}{s^2} - \frac{14}{s}\right) + e^{-5s}\left(-\frac{2}{s^2} - \frac{11}{s}\right)$$

Inverse Laplace Transforms with step functions.

Example.

$$L^{-1}\left(\frac{e^{-2s}}{s^2}\right) = L^{-1}\left(e^{-2s}\left(\frac{1}{s^2}\right)\right)$$

$u_c(t)f(t-c)$	$e^{-cs}F(s)$

$$c = 2, F(s) = \frac{1}{s^2}$$

$$f(t) = t$$

$$L^{-1}\left(\frac{e^{-2s}}{s^2}\right) = u_2(t)(t-2)$$

Or equivalently

$$L^{-1}\left(\frac{e^{-2s}}{s^2}\right) = u(t-2)(t-2)$$

Factor out the e to get the jump point (c value). Do the inverse of the function multiplying the e term. Then apply the shift to write the final function.

Example.

$$F(s) = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{2}{s}\right) + e^{-4s}\left(\frac{4}{s^3} + \frac{1}{s}\right)$$

Find the inverse Laplace transform of F(s).

$$f(t) = t - u_2(t)(t+1) + u_4(t)[\,2(t-4)^2 + 1]$$

The jump point for the first piece is c=1,

$$g(t) = t + 2 \rightarrow g(t-1) = (t-1) + 2 = t + 1$$

The jump point for the second piece is c=4

$$h(t) = 2t^2 + 1 \rightarrow h(t-4) = 2(t-4)^2 + 1$$

Dirac Delta Function

$$\delta(t) = \begin{cases} 1, t = 0, \\ 0, otherwise \end{cases}$$

Impulse function

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

In this course, if you get terms in F(s) that have no s or have an s in the numerator larger than the denominator: this is an indication of an arithmetic mistake, not a delta function.

Convolutions.

$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$

A convolution in this context is a function defined by an integral with the product of two functions. One of which represents a shift and is the dummy variable in the integration.

Example in forward direction.
Find the Laplace Transform of the convolution function:

$$h(t) = \int_0^t (t-\tau)^2 \sin(\tau)\, d\tau$$

Identify the two functions to be transformed.

$$f(t) = t^2, g(t) = \sin(t)$$

$$F(s) = \frac{2}{s^3}, G(s) = \frac{1}{s^2+1}$$

$$H(s) = \frac{2}{s^3(s^2+1)}$$

Find the inverse Laplace transform of using convolutions:

$$H(s) = \frac{s}{(s+2)(s^2+9)}$$

The old approach: apply partial fractions to split this into two (or three) terms.

$$\frac{A}{s+2} + \frac{Bs+C}{s^2+9}$$

Here, instead, we need to break this into a product of two expressions each of which we can apply the inverse Laplace transform to.

This won't work:

$$\left(\frac{s}{s+2}\right)\left(\frac{1}{s^2+9}\right)$$

Instead:

$$\left(\frac{1}{s+2}\right)\left(\frac{s}{s^2+9}\right)$$

$$f(t) = e^{-2t}, g(t) = \cos(3t)$$

$$h(t) = \int_0^t e^{-2(t-\tau)}\cos(3\tau)\, d\tau$$

Example with initial value problem (piecewise function)

$$y'' + y = \begin{cases} e^{2t}, 0 \le t < 2 \\ 1, \quad\quad t \ge 2 \end{cases}, y(0) = 3, y'(0) = -1$$

$$f(t) = \begin{cases} e^{2t}, 0 \le t < 2 \\ 1, \quad\quad t \ge 2 \end{cases}$$

$$f(t) = e^{2t} + \begin{cases} 0, & 0 \le t < 2 \\ 1 - e^{2t}, & t \ge 2 \end{cases}$$

$$f(t) = e^{2t} + (1 - e^{2t})u_2(t)$$

$$y'' + y = e^{2t} + (1 - e^{2t})u_2(t)$$

$\boldsymbol{f'(t)}$	$sF(s) - f(0)$
$\boldsymbol{f''(t)}$	$s^2F(s) - sf(0)$ $- f'(0)$

$\boldsymbol{u_c(t)g(t)}$	$e^{-cs}\mathfrak{L}\{g(t+c)\}$

$$s^2Y(s) - s(3) - (-1) + Y(s) = \frac{1}{s-2} + e^{-2s}\left(\frac{1}{s} + \frac{e^4}{s-2}\right)$$

$$g(t) = 1 - e^{2t} \rightarrow g(t+2) = 1 - e^{2(t+2)} = 1 - e^{2t+4} = 1 - (e^4)e^{2t}$$

$$s^2Y(s) + Y(s) = \frac{1}{s-2} + e^{-2s}\left(\frac{1}{s} + \frac{e^4}{s-2}\right) + 3s - 1$$

$$Y(s)(s^2+1) = e^{-2s}\left(\frac{1}{s} + \frac{e^4}{s-2}\right) + \frac{1}{s-2} + \frac{(3s-1)(s-2)}{s-2}$$

$$Y(s)(s^2+1) = e^{-2s}\left(\frac{1}{s}+\frac{e^4}{s-2}\right)+\frac{3s^2-7s+3}{s-2}$$

$$Y(s) = e^{-2s}\left(\frac{1}{s}+\frac{e^4}{s-2}\right)\left(\frac{1}{s^2+1}\right)+\frac{3s^2-7s+3}{(s-2)(s^2+1)}$$

$$Y(s) = e^{-2s}\left(\frac{s-2}{s(s-2)}+\frac{se^4}{s(s-2)}\right)\left(\frac{1}{s^2+1}\right)+\frac{3s^2-7s+3}{(s-2)(s^2+1)}$$

$$Y(s) = e^{-2s}\left(\frac{s(1+e^4)-2}{s(s-2)(s^2+1)}\right)+\frac{3s^2-7s+3}{(s-2)(s^2+1)}$$

$$L^{-1}\left(\frac{3s^2-7s+3}{(s-2)(s^2+1)}\right)$$

$$\frac{A}{s-2}+\frac{Bs+C}{s^2+1} = \frac{(As^2+A+Bs^2-2Bs+Cs-2C)}{(s-2)(s^2+1)}$$

$$A+B = 3\ (s^2)$$
$$2B+C = -7\ \ (s)$$
$$A-2C = 3\ \ (1)$$

$$A = \frac{23}{3}, B = -\frac{14}{3}, C = \frac{7}{3}$$

$$\frac{\frac{23}{3}}{s-2}+\frac{\left(-\frac{14}{3}s\right)}{s^2+1}+\frac{\left(\frac{7}{3}\right)}{s^2+1}$$

$$L^{-1}\left(\frac{\frac{23}{3}}{s-2}+\frac{\left(-\frac{14}{3}s\right)}{s^2+1}+\frac{\left(\frac{7}{3}\right)}{s^2+1}\right) = \frac{23}{3}e^{2t}-\frac{14}{3}\cos t+\frac{7}{3}\sin t$$

This is the piece before the step is applied. Now, the piece afterwards:

$$L^{-1}\left(e^{-2s}\left(\frac{s(1+e^4)-2}{s(s-2)(s^2+1)}\right)\right)$$

$$c = 2, G(s) = \frac{s(1+e^4)-2}{s(s-2)(s^2+1)}$$

$$\frac{A}{s}+\frac{B}{s-2}+\frac{Cs+D}{s^2+1} = \frac{\left(A(s-2)(s^2+1)+Bs(s^2+1)+(Cs+D)s(s-2)\right)}{s(s-2)(s^2+1)}$$

$$A(s^3 - 2s^2 + s - 2) + Bs^3 + Bs + Cs^3 - 2Cs^2 + Ds^2 - 2Ds = s(1 + e^4) - 2$$

$$A + B + C = 0 \ \ (s^3)$$
$$-2A - 2C + D = 0 \ (s^2)$$
$$A + B - 2D = (1 + e^4) \ \ (s)$$
$$-2A = -2 \ \ (1)$$

$$A = 1,$$

$$1 + B + C = 0 \ \ (s^3)$$
$$-2 - 2C + D = 0 \ (s^2)$$
$$1 + B - 2D = (1 + e^4) \ \ (s)$$

$$B + C = -1 \ \ (s^3)$$
$$-2C + D = 2 \ (s^2)$$
$$B - 2D = e^4 \ \ (s)$$

$$B + C = -1 \ \ (s^3)$$
$$-B + 2D = -e^4 \ \ (s)$$

$$C + 2D = 1 - e^4$$

$$-2C + D = 2 \ (s^2)$$
$$2C + 4D = 2 - 2e^4$$

$$5D = 4 - 2e^4$$

$$D = \frac{4 - 2e^4}{5}$$

$$C = -2\left(\frac{4 - 2e^4}{5}\right) + 1 - e^4 = -\frac{8}{5} + \frac{4e^4}{5} + 1 - e^4 = -\frac{3}{5} - \frac{1}{5}e^4$$

$$B = \frac{3}{5} + \frac{1}{5}e^4 - 1 = -\frac{2}{5} + \frac{1}{5}e^4$$

$$G(s) = \frac{1}{s} + \frac{\left(-\frac{2}{5} + \frac{1}{5}e^4\right)}{s - 2} + \frac{\left(-\frac{3}{5} - \frac{1}{5}e^4\right)s}{s^2 + 1} + \frac{\left(\frac{4 - 2e^4}{5}\right)}{s^2 + 1}$$

$$g(t) = 1 + \left(-\frac{2}{5} + \frac{1}{5}e^4\right)(e^{2t}) + \left(-\frac{3}{5} - \frac{1}{5}e^4\right)\cos(t) + \left(\frac{4 - 2e^4}{5}\right)\sin(t)$$

$\boldsymbol{u_c(t)f(t-c)}$	$e^{-cs}F(s)$

$$\rightarrow u_2(t)\left[1+\left(-\frac{2}{5}+\frac{1}{5}e^4\right)\left(e^{2(t-2)}\right)+\left(-\frac{3}{5}-\frac{1}{5}e^4\right)\cos(t-2)+\left(\frac{4-2e^4}{5}\right)\sin(t-2)\right]$$

The final solution combines these pieces.

$$\begin{aligned} y(t) = \frac{23}{3}e^{2t} - \frac{14}{3}\cos t + \frac{7}{3}\sin t \\ + u_2(t)\left[1+\left(-\frac{2}{5}+\frac{1}{5}e^4\right)\left(e^{2(t-2)}\right)+\left(-\frac{3}{5}-\frac{1}{5}e^4\right)\cos(t-2)\right. \\ \left. + \left(\frac{4-2e^4}{5}\right)\sin(t-2)\right] \end{aligned}$$

This is the end of Laplace transforms.
Next time, we will briefly discuss systems of equations, and their connection to second-order problems.