9/12/2022

Introduction to the course
Types of DEs
Direction Fields

Differential Equations are equations that contain derivatives.

$$f'(x) = x^2 + 1$$

Constants of integration will be in the solution unless there is information about the solution at a point in time/space so that the constant can be solved for. Initial value problems (IVP). (Typically, these occur when x=0 or t=0, but if they are not zero, sometimes they are referred as boundary value problems (BVP).) These are basically the same thing when we have only one variable, and only one derivative (first derivative). The difference will matter only when we have second derivatives or when we have more than one variable.

Ordinary differential equations (ODE) contain the derivative for functions that have only one independent variable. Partial differential equations (PDE) contain derivatives for functions that depend on more than one independent variable.

ODEs

$$f'(x) = x^2 + 1$$

$$\frac{df}{dx} = kf + x$$

$$\frac{d^2f}{dx^2} = kf + x^2$$

$$y'' + y' + y = 0$$

PDEs

$$f_{xx} - f_{yy} = 0$$

$$\frac{\partial f}{\partial x}(x^2) + \frac{\partial^2 f}{\partial x \partial y} = xy$$

Order of a differential equation:
Is based on the highest derivative in the equation.
First derivative = first order

$$f'(x) = x^2 + 1$$

$$\frac{df}{dx} = kf + x$$

$$\frac{\partial f}{\partial x} = xy$$

Second order

$$\frac{d^2 f}{dx^2} = xf + x^2$$

$$y'' + y' + y = 0$$

$$f_{xx} - f_{yy} = 0$$

$$\frac{\partial f}{\partial x}(x^2) + \frac{\partial^2 f}{\partial x \partial y} = xy$$

Third order

$$y''' + y' - y = 0$$

Fourth order

$$y^{IV} - y = 0$$
$$y^{(4)} - y = 0$$

Degree of a differential equation:
Is related to the highest power of the derivatives

Linear vs. non-linear
A linear differential is linear in the function and its derivatives, but need not be linear in the independent variable(s).

Linear:

$$y'' + y' + y = 0$$
$$x^2 y'' + xy' + y = 0$$

Nonlinear:

$$y'' + y' + y = \cos(y)$$
$$\left(\frac{dy}{dx}\right)^2 + y = 0$$

$$y''(y') = \frac{x^2}{2}$$

Direction Fields/Slope Fields
A way of visualizing the behavior of solutions of a differential equation (first order) without solving it analytically (without finding an equation that solves the problem).

Start with something relatively simple.

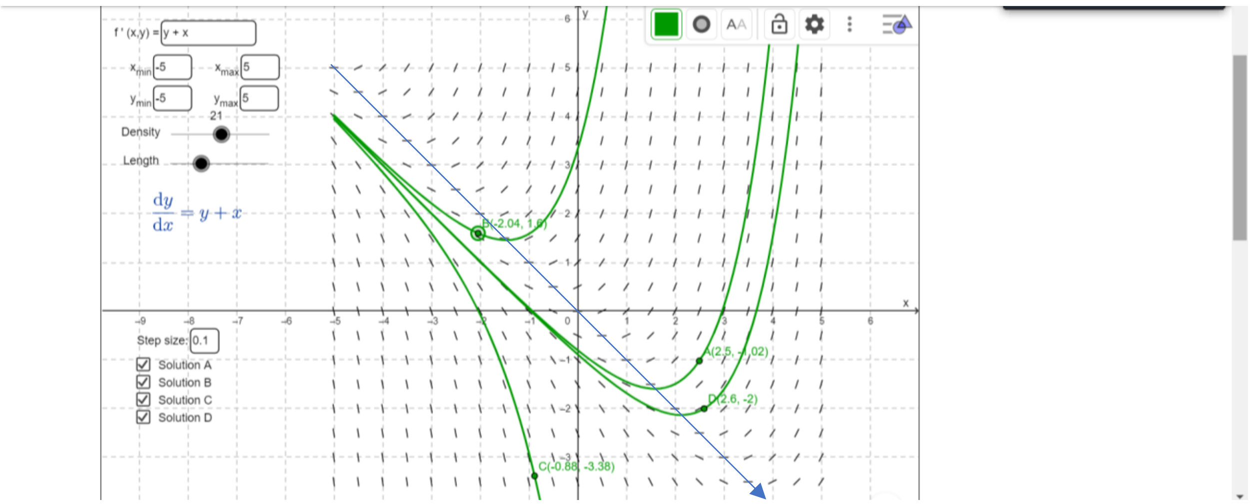
$$y' = y + 2$$

Recall from calc I that y' is the slope of the tangent line at a point.
This ODE does not depend on any independent variable (like t), so it doesn't change over time.
(Autonomous differential equations do not depend on the independent variable. The derivative only depends on the value of function.)

Suppose I know that I'm starting out at y=0, then the ODE tells that slope is $y' = 2$. Picture a slope field as a grid of slope values at points on the graph. (x,0) and plot a slope that has a value of 2.

A value of y where the slope is 0 is called an equilibrium. In this equation y=-2 is an equilibrium.

A nullcline is a function or curve where the derivative is zero (rather than a constant). It is not an equilibrium since you can't stay on the nullcline as you move in x or time.



Solving differential equations using separation of variables or regular integration.

Solving equations that have a derivative and no other function terms.

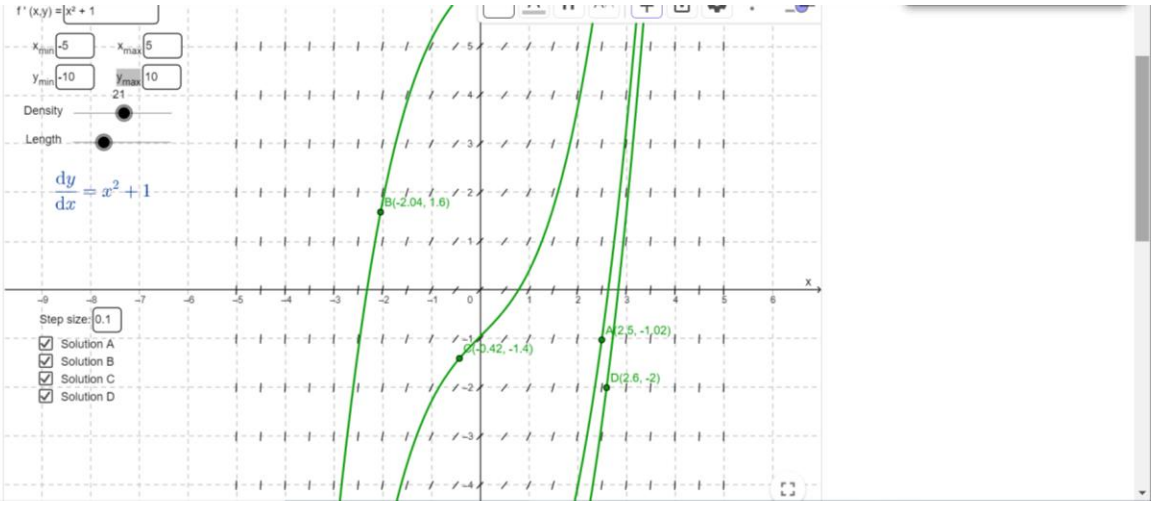
$$f'(x) = x^2 + 1$$

If the equation contains no other functions, then put the derivative on one side and integrate in the remaining variable.

$$f(x) = \int x^2 + 1dx = \frac{x^3}{3} + x + C$$

The function that satisfies our differential is $f(x) = \frac{1}{3}x^3 + x + C$.

This is a family of solutions. The family will all be shaped the same way, but will have a vertical shift to the solutions.



These are all the same cubic functions, but shifted up or down.

If we have an initial condition, such as $f(0) = 1$, then we have enough information to find the specific solution that satisfies both the differential equation and the initial condition.

$$\frac{1}{3}(0)^3 + 0 + C = 1$$
$$C = 1$$

$$f(x) = \frac{1}{3}x^3 + x + 1$$

$$y''(x) = e^x + x$$

$$y'(x) = \int e^x + x dx = e^x + \frac{1}{2}x^2 + C_1$$

$$y(x) = \int e^x + \frac{1}{2}x^2 + C_1 dx = e^x + \frac{1}{6}x^3 + C_1 x + C_2$$

To solve for both of these constants, I need two conditions. Either two initial conditions, say $y(0) = 1$ and $y'(0) = 0$, or two boundary conditions, say $y(0) = 1, y(1) = 1$.

Let's solve for the initial conditions.

$$e^0 + \frac{1}{2}0^2 + C_1 = 0$$

$$1 + C_1 = 0$$
$$C_1 = -1$$

$$y(x) = e^x + \frac{1}{6}x^3 - x + C_2$$
$$e^0 + \frac{1}{6}0^3 - 0 + C_2 = 1$$

$$1 + C_2 = 1$$
$$C_2 = 0$$

$$y(x) = e^x + \frac{1}{6}x^3 - x$$

Simplest kind of differential equations. Things you have encountered in calculus.

We want to be able to extend our solutions to equations that contain both the function and its derivative.

$$y' = y + 2$$

Separation of variables.

$$\frac{dy}{dx} = y + 2$$

Rearrange the equation to put all the y variables on the left side and all the x variables on the right side. We can't add and subtract. Want to be able to use the dy and dx as signals of integration with respect to the corresponding variable. Something akin to the "reverse" chain rule.

Multiply on both sides of the equation by dx, and going to divide by (y+2).

$$\frac{dy}{y+2} = dx$$

Integrate each side of the equation with the respective variables: integrate with y on the left, and integrate with x on the right.

$$\int \frac{dy}{y+2} = \int dx$$

$$\ln|y + 2| = x + C$$

$$e^{\ln|y+2|} = e^{x+C}$$
$$y + 2 = e^x e^C$$

Let $e^c = A$

$$y + 2 = Ae^x$$
$$y = Ae^x - 2$$

I would need an initial condition to solve for my constant A.

But does it satisfy the initial equation?

$$y' = y + 2$$

$$y' = Ae^x = Ae^x - 2 + 2 = Ae^x$$

Our solution method does work for the family of solutions.

Separation of variables works when we can algebraically separate the variables with all y's on one side and all independent variables on the other side.

What can a separable equation look like before we get started?

$$y' = f(x, y)$$

Separable

$$y' = xy$$

Non-separable

$$y' = x + y$$

$$M(y)y' = N(x)$$

$$M(x, y)dy + N(x, y)dx = 0$$

In these cases you have the most algebra to do, and whether or not it can be separated is going to depend of the specifics of what M(x,y) and N(x,y) are.

Separable

$$e^x y\, dy + x\sqrt{1 - y^2}dx = 0$$
$$e^x ydy = -x\sqrt{1 - y^2}dx$$

$$\frac{ydy}{\sqrt{1 - y^2}} = -xe^{-x}dx$$

I could integrate this. Left side would use u-substitution. The right side would use integration by parts.

Not separable

$$(x + y)dy + (e^{xy})dx = 0$$

I will do my best to avoid integrating with trig substitution. I will limit the partial fractions cases. When we look at logistic functions later (population modeling problems, automonous) we will require partial fractions.

Continue solving our example from above.

$$\frac{ydy}{\sqrt{1 - y^2}} = -xe^{-x}dx$$

$$\int \frac{ydy}{\sqrt{1 - y^2}}$$
$$u = 1 - y^2$$
$$du = -2ydy$$

$$-\frac{1}{2}du = ydy$$

$$\int \frac{ydy}{\sqrt{1-y^2}} = \int \frac{\frac{1}{2}du}{u^{\frac{1}{2}}} = \int \frac{1}{2}u^{-\frac{1}{2}}du = u^{\frac{1}{2}}\frac{1}{2}(2) = \sqrt{u} = \sqrt{1-y^2}$$

$$\int -xe^{-x}dx$$

$$u = -x, dv = e^{-x}dx$$
$$du = -1dx, v = -e^{-x}$$

$$\int -xe^{-x}dx = xe^{-x} - \int e^{-x}\,dx = xe^{-x} + e^{-x} + C$$

$$\sqrt{1-y^2} = xe^{-x} + e^{-x} + C$$

It is okay to leave messing problems in implicit form when the explicit form will introduce square roots or be uglier than what we start with.

Example.

$$\frac{dy}{dt} = e^{y-t}\sec(y)\,(1+t^2), y(0) = 0$$

$$\frac{dy}{dt} = e^{y}e^{-t}\sec(y)\,(1+t^2)$$

$$\frac{e^{-y}dy}{\sec(y)} = e^{-t}(1+t^2)dt$$

$$\cos(y)\,e^{-y}dy = (1+t^2)e^{-t}dt$$

$$\int e^{-y}\cos(y)\,dy =$$

$$u = \cos(y)\,, dv = e^{-y}dy$$
$$du = -\sin(y)\,dy, v = -e^{-y}$$

$$-e^{-y}\cos(y) - \int e^{-y}\sin(y)\,dy$$

$$u = \sin(y)\,, dv = e^{-y}dy$$
$$du = \cos(y)\,dy, v = -e^{-y}$$

$$-e^{-y}\cos(y) - \left[-e^{-y}\sin(y) - \int -e^{-y}\cos(y)\,dy\right]$$
$$-e^{-y}\cos(y) + \left[e^{-y}\sin(y) - \int e^{-y}\cos(y)\,dy\right] = \int e^{-y}\cos(y)\,dy$$

$$-e^{-y}\cos(y) + e^{-y}\sin(y) = 2\int e^{-y}\cos(y)\,dy$$

$$\int e^{-y}\cos(y)\,dy = \frac{1}{2}\left(e^{-y}\sin(y) - e^{-y}\cos(y)\right) + C$$

$$\int (1+t^2)e^{-t}dt$$

+/-	u	dv
+	$1+t^2$	e^{-t}
-	$2t$	$-e^{-t}$
+	2	e^{-t}
-	0	$-e^{-t}$

$$\int (1+t^2)e^{-t}dt = (1+t^2)(-e^{-t}) - 2t(e^{-t}) + (2)(-e^{-t}) + C$$

$$\frac{1}{2}\left(e^{-y}\sin(y) - e^{-y}\cos(y)\right) = -(1+t^2)(e^{-t}) - 2t(e^{-t}) - (2)(e^{-t}) + C$$

$$\frac{1}{2}\left(e^{-0}\sin(0) - e^{-0}\cos(0)\right) = -(1+0^2)(e^{-0}) - 2(0)(e^{-0}) - (2)(e^{-0}) + C$$

$$\frac{1}{2}(1) = -(1)(1) - (2)(1) + C$$

$$\frac{1}{2} = -1 - 2 + C$$

$$\frac{1}{2} = -3 + C$$

$$\frac{7}{2} = C$$

$$\frac{1}{2}\left(e^{-y}\sin(y) - e^{-y}\cos(y)\right) = -(1+t^2)(e^{-t}) - 2t(e^{-t}) - (2)(e^{-t}) + \frac{7}{2}$$

Verifying that an equation is a solution to a differential equation.

Suppose we have a differential equation like

$$y'' + 4y' - 12y = 0$$

I propose a solution of $y(t) = c_1 e^{-6t}$
Verify the function is a solution to the differential equation.

$$y' = -6c_1e^{-6t}$$
$$y'' = 36c_1e^{-6t}$$
$$y'' + 4y' - 12y = 0$$

$$36c_1e^{-6t} + 4(-6c_1e^{-6t}) - 12c_1e^{-6t} =?= 0$$

$$c_1(e^{-6t})(36 - 24 - 12) = 0$$

This means that this solution does satisfy the equation. If the equation is not equal on both sides, then or the function is not a solution to the equation.

There is a problem on the written homework that requires you to know hyperbolic trig functions.