10/19/2023

Power Series: Convergence, Intervals, Radii
Writing Functions as Power Series: geometric series, center (completing the square, recentering), integration, derivatives

Power Series
Infinite series that involve powers of x or (x-c)

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$$

Does the power series converge and if so, for what values of x does it converge?

Generally, use the ratio test to obtain a condition on convergence.

$$\lim_{n\to\infty} \left| \frac{2^{n+1} x^{n+1}}{(n+1)!} \times \frac{n!}{2^n x^n} \right| = \lim_{n\to\infty} \left| \frac{2x}{n+1} \right| = 0 < 1$$

Since our ratio test left us with a condition that does not depend on x, we can say that the power series converges for all real numbers.

Interval of convergence, and in this case, the interval of convergence is $(-\infty, \infty)$. Related to that is the radius of convergence: the distance from the center to either end of the interval, or half the length of the interval.

In general, if the interval of convergence is (a, b) (closed or half-closed intervals are treated the same), then the radius of convergence is $\frac{b-a}{2}$. So, here the radius is ∞.

Consider

$$\sum_{n=1}^{\infty} \frac{n x^n}{3^n}$$

What is the interval of convergence? What is the radius of convergence?

$$\lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{3^{n+1}} \times \frac{3^n}{n x^n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \times \frac{x}{3} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \right| \times \lim_{n\to\infty} \left| \frac{x}{3} \right| = (1) \left| \frac{x}{3} \right| < 1$$

$$\left| \frac{x}{3} \right| < 1$$

$$-1 < \frac{x}{3} < 1$$

$$-3 < x < 3$$

If the interval is finite, then check both endpoints to see if either or both should be included.

Checking $x = -3$

$$\sum_{n=1}^{\infty} \frac{nx^n}{3^n} \to \sum_{n=1}^{\infty} \frac{n(-3)^n}{3^n} = \sum_{n=1}^{\infty} \frac{n(-1)^n 3^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n n$$

Diverges by the divergence test since the limit of n is not 0.

Checking $x = 3$

$$\sum_{n=1}^{\infty} \frac{nx^n}{3^n} \to \sum_{n=1}^{\infty} \frac{n(3)^n}{3^n} = \sum_{n=1}^{\infty} n$$

This also diverges by the divergence test.

The interval of convergence is indeed $(-3,3)$.

The radius of convergence is $\frac{3-(-3)}{2} = \frac{6}{2} = 3$

Compare with

$$\sum_{n=1}^{\infty} \frac{x^n}{n3^n} \; or \; \sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)3^{n+1}} \times \frac{n3^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \times \frac{x}{3} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \right| \times \lim_{n\to\infty} \left| \frac{x}{3} \right| = (1) \left| \frac{x}{3} \right| < 1$$

$$\left| \frac{x}{3} \right| < 1$$

$$-1 < \frac{x}{3} < 1$$

$$-3 < x < 3$$

Checking $x = -3$

$$\sum_{n=1}^{\infty} \frac{x^n}{n3^n} \to \sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

This is the alternating harmonic series. Since the limit of 1/n goes to 0, this endpoint converges.

Checking $x = 3$

$$\sum_{n=1}^{\infty} \frac{x^n}{n3^n} \rightarrow \sum_{n=1}^{\infty} \frac{(3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

This endpoint diverges by the p-series test.

My interval of convergence is $[-3,3)$

The radius of convergence is still R=3.

Vs. $\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^2 3^{n+1}} \times \frac{n^2 3^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{n^2}{(n+1)^2} \times \frac{x}{3} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \right|^2 \times \lim_{n\to\infty} \left| \frac{x}{3} \right| = (1)^2 \left| \frac{x}{3} \right| < 1$$

$$\left| \frac{x}{3} \right| < 1$$

$$-1 < \frac{x}{3} < 1$$

$$-3 < x < 3$$

Checking $x = -3$

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n} \rightarrow \sum_{n=1}^{\infty} \frac{(-3)^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

This is the alternating p-series. Since the limit of $\frac{1}{n^2}$ goes to 0, this endpoint converges.

Checking $x = 3$

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n} \rightarrow \sum_{n=1}^{\infty} \frac{(3)^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

This endpoint converges by the p-series test.

My interval of convergence is $[-3,3]$

The radius of convergence is still R=3.

Power series may also only converge at a single point.

$$\sum_{n=1}^{\infty} \frac{n!\, x^n}{10^n}$$

$$\lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{10^{n+1}}\times\frac{10^n}{n!\,x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x}{10}\right| = \infty$$

Unless $x = 0$.

When $x = 0$, then every term in the series is zero, and so the sum is zero. This will diverge everywhere except when x=0.

Interval of convergence is {0}, and the radius of convergence is also 0.

Example of center not at 0.

$$\sum_{n=0}^{\infty}\frac{(x-2)^n}{6^n}$$

$$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{6^{n+1}}\times\left(\frac{6^n}{(x-2)^n}\right)\right| = \lim_{n\to\infty}\left|\frac{x-2}{6}\right| < 1$$

$$\left|\frac{x-2}{6}\right| < 1$$

$$-1 < \frac{x-2}{6} < 1$$

$$-6 < x-2 < 6$$

$$-4 < x < 8$$

Checking $x = -4$

$$\sum_{n=0}^{\infty}\frac{(x-2)^n}{6^n} \to \sum_{n=0}^{\infty}\frac{(-4-2)^n}{6^n} = \sum_{n=0}^{\infty}\frac{(-6)^n}{6^n} = \sum_{n=0}^{\infty}(-1)^n$$

Diverges by divergence test

Checking $x = 8$

$$\sum_{n=0}^{\infty}\frac{(x-2)^n}{6^n} \to \sum_{n=0}^{\infty}\frac{(8-2)^n}{6^n} = \sum_{n=0}^{\infty}\frac{(6)^n}{6^n} = \sum_{n=0}^{\infty}1$$

Diverges by the divergence test

So the interval of convergence is $(-4,8)$. And the radius of convergence $R = \frac{8-(-4)}{2} = \frac{12}{2} = 6$

Write functions as power series
Recall the geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

When the series converges.

We can turn this into a power series formula by replacing r with x, or more generally, g(x).

$$\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$$

Consider the function $f(x) = \frac{2x}{1-x}$. In this situation $r = x, a = 2x$

$$\frac{2x}{1-x} = \sum_{n=0}^{\infty} (2x)(x)^n = \sum_{n=0}^{\infty} 2x^{n+1}$$

Interval of convergence:

$$\lim_{n\to\infty} \left|\frac{2x^{n+2}}{2x^{n+1}}\right| = \lim_{n\to\infty} |x| < 1$$
$$(-1,1)$$

Radius of convergence is 1.

Example.

$$f(x) = \frac{4}{3-2x}$$

Write a power series for this function. (unless otherwise specified, the center is at x=0)
Step 1: turn the constant in the denominator into a 1

$$f(x) = \frac{4}{3-2x} \times \frac{\frac{1}{3}}{\frac{1}{3}} = \frac{\left(\frac{4}{3}\right)}{1-\frac{2}{3}x}$$

$$r = \frac{2}{3}x, a = \frac{4}{3}$$

$$\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)\left(\frac{2}{3}x\right)^n = \sum_{n=0}^{\infty} \left(\frac{4}{3}\right)\left(\frac{2}{3}\right)^n x^n = \sum_{n=0}^{\infty} \frac{2^2 2^n x^n}{3(3^n)} = \sum_{n=0}^{\infty} \frac{2^{n+2} x^n}{3^{n+1}}$$

Find a power series for the function $f(x) = \frac{3}{1-2x}$ centered at x=4

Rewrite 1-2x in the form of a-b(x-4)

$$1 - 2x = a - b(x - 4) = a - bx + 4b$$

$$-b = -2, b = 2$$

$$a + 4b = 1, a + 4(2) = 1, a + 8 = 1, a = -7$$

$$-7 - 2(x - 4) = -7 - 2x + 8 = 1 - 2x$$

$$f(x) = \frac{3}{-7 - 2(x - 4)} \times \frac{\left(-\frac{1}{7}\right)}{\left(-\frac{1}{7}\right)} = \frac{-\frac{3}{7}}{1 + \frac{2}{7}(x - 4)} = \frac{-\frac{3}{7}}{1 - \left[-\frac{2}{7}(x - 4)\right]}$$

$$r = -\frac{2}{7}(x - 4), a = -\frac{3}{7}$$

$$\sum_{n=0}^{\infty} -\frac{3}{7}\left(-\frac{2}{7}(x - 4)\right)^n = \sum_{n=0}^{\infty} -\frac{3}{7}\left(-\frac{2}{7}\right)^n (x - 4)^n = \sum_{n=0}^{\infty} \frac{3(-1)^{n+1} 2^n (x - 4)^n}{7^{n+1}}$$

If you have x anywhere in the expression, those also have to be shifted to be of the form (x-c) like the denominator.

Two functions have rational expressions as their derivatives:

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

$$\frac{d}{dx}[\arctan x] = \frac{1}{1 + x^2}$$

To do natural log is similar to arctangent except that you need to shift the center from 0, since neither the function nor the derivative is defined there. Typically, the shift is to center at x=1.

You would find a power series for $\frac{1}{x-1+1} = \frac{1}{1+(x-1)}, r = -(x - 1), a = 1$

For arctangent, start with $f'(x) = \frac{1}{1+x^2}, r = -x^2, a = 1$

$$f'(x) = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Find the antiderivative to get back to the original arctangent function

$$\int \sum_{n=0}^{\infty} (-1)^n x^{2n}\, dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n + 1}$$

This is the power series for $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

Consider the function $f(x) = \frac{1}{x^2+6x+13}$

There is a tendency for students to rewrite this as $\frac{1}{13+6x+x^2}$ and divide everything by 13 to get $\frac{\frac{1}{13}}{1+\frac{6}{13}x+\frac{x^2}{13}}$

Then try to say $r = \left(-\frac{6}{13}x - \frac{x^2}{13}\right), a = \frac{1}{13}$

You can't do this. The expression for r must be of the form kx^p or $k(x-c)$.

They can't have multiple x's.

The way to tackle this kind of problem instead is to complete the square.

$$\frac{1}{x^2+6x+13} = \frac{1}{x^2+6x+9+4} = \frac{1}{4+(x+3)^2}$$

This expression is centered naturally at x= -3.

Derivatives of the power series formula.

$$a(1-r)^{-1} = \frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$$

Take a derivative, get a new formula. (with respect to r)

$$a(1-r)^{-2}(-1)(-1) = a(1-r)^{-2} = \frac{a}{(1-r)^2} = \sum_{n=1}^{\infty} anr^{n-1} = \sum_{n=0}^{\infty} a(n+1)r^n$$

$$\frac{a}{(1-r)^2} = \sum_{n=0}^{\infty} a(n+1)r^n$$

$$f(x) = \frac{x^4}{(1-3x)^2} = \sum_{n=0}^{\infty} x^4(3x)^n(n+1) = \sum_{n=0}^{\infty} (n+1)3^n x^{n+4}$$

Another derivative:

$$a(1-r)^{-2} = \frac{a}{(1-r)^2} = \sum_{n=1}^{\infty} anr^{n-1}$$

$$a(-2)(1-r)^{-3}(-1) = \frac{2a}{(1-r)^3} = \sum_{n=2}^{\infty} an(n-1)r^{n-2} = \sum_{n=0}^{\infty} a(n+2)(n+1)r^n$$

$$f(x) = \frac{x^2}{(5-4x)^3} \times \frac{\frac{1}{5^3}}{\frac{1}{5^3}} = \frac{\left(\frac{x^2}{125}\right)}{\left(1-\frac{4}{5}x\right)^3}, r = \frac{4}{5}x, a = \frac{1}{125}x^2$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{125}\right) x^2 (n+2)(n+1)\left(\frac{4}{5}x\right)^n = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)4^n x^{n+2}}{5^{n+3}}$$

And of course, you can keep going.

Next time we will start talking about Taylor Series (Maclaurin series)