10/26/2023

Taylor series errors
Doing division with Taylor series
Using Taylor series for limits

Review for Exam #2 (Tuesday)

Taylor series errors: $R_n \leq \frac{\left|\max_{x \in I} f^{(n+1)}(z)\right|}{(n+1)!}(x-a)^{(n+1)}$

x is the point at which we are doing our approximation, and I is an interval that contains both a and x. It may be a global maximum, or a may be a maximum on a finite interval.

Find the Taylor series (Maclaurin series) for $f(x) = \sin(x)$, centered at a=0 for n=4

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$(x-a)^n$	$\frac{f^{(n)}(a)(x-a)^n}{n!}$
0	1	$\sin(x)$	0	1	0
1	1	$\cos(x)$	1	x	x
2	2	$-\sin(x)$	0	x^2	0
3	6	$-\cos(x)$	-1	x^3	$-\frac{x^3}{6}$
4	24	$\sin(x)$	0	x^4	0
5	120	$\cos(x)$	1	x^5	
6	720				

$$P_4 = x - \frac{x^3}{6} + R_4$$

$$R_4 = \leq \frac{\left|\max_{x \in I} f^{(5)}(z)\right|}{(5)!}(x)^{(5)}$$

Suppose I want to approximate the value of $\sin\left(\frac{1}{2}\right)$ (in radians). What is the approximation and the estimated error?

$$P_4\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right) - \frac{\left(\frac{1}{2}\right)^3}{6} = \frac{23}{48} \approx 0.4791 \ldots$$

$$Error \leq \frac{1}{5!}\left(\frac{1}{2}\right)^5 = 2.60416 \ldots \times 10^{-4} = 0.0002604 \ldots$$

Estimate the value of $f(x) = e^x$ evaluated at x=1/2, for the Maclaurin polynomial of n=5. Estimate the error.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$(x-a)^n$	$\frac{f^{(n)}(a)(x-a)^n}{n!}$
0	1	e^x	1	1	1
1	1	e^x	1	x	x
2	2	e^x	1	x^2	$\frac{x^2}{2}$
3	6	e^x	1	x^3	$\frac{x^3}{6}$
4	24	e^x	1	x^4	$\frac{x^4}{24}$
5	120	e^x	1	x^5	$\frac{x^5}{120}$
6	720	e^x	1		

$$P_5 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + R_5$$

$$R_5 \le \frac{\left|\max_{x \in I} e^x\right|}{(6)!}(x)^{(6)} = \frac{\left(e^{\frac{1}{2}}\right)\left(\frac{1}{2}\right)^6}{6!} = 5.899 \ldots \times 10^{-5}$$

If the center is 0, and we are approximating at either -1/2 or ½, use the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$

Division with Taylor series

Traditionally for finite polynomials we start with the highest degree terms and work down to end with a proper fraction, all terms in descending order. But here, we start with the lowest degree terms and the higher terms trail off into infinity (in ascending order).

Find a power series for $f(x) = \frac{\sin x}{x+1}$

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

See next page…

$$x - x^2 + \frac{5}{6}x^3 - \frac{5}{6}x^4 \cdots$$

$$1+x \;\overline{)\,0 + x + 0x^2 - \frac{1}{6}x^3 + 0x^4 + \frac{1}{120}x^5 + \cdots}$$

$$-x \mp x^2$$

$$-x^2 - \frac{1}{6}x^3$$

$$+x^2 + x^3$$

$$\frac{5}{6}x^3 + 0x^4$$

$$-\frac{5}{6}x^3 \mp \frac{5}{6}x^4$$

$$-\frac{5}{6}x^4 + \frac{1}{120}x^5$$

$$f(x) = \frac{\sin(x)}{x+1} \approx x - x^2 + \frac{5}{6}x^3 - \frac{5}{6}x^4 + \cdots$$

Example of using Taylor series for limits:

$$\lim_{x\to 0}\frac{\cos x - 1}{x^2} = \lim_{x\to 0}\frac{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} - 1}{x^2} = \lim_{x\to 0}\frac{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots - 1}{x^2} =$$

$$\lim_{x\to 0}\frac{-\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots}{x^2} = \lim_{x\to 0} -\frac{1}{2} + \frac{x^2}{24} - \frac{x^4}{720} + \cdots = -\frac{1}{2}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}\right)$$

End material for Exam #2

Sequences!!!!
Series Tests
Power Series
Taylor Series/Maclaurin
Applications with Power series: adding, multiplying, dividing, finding limits, integrating/differentiating, etc.

Be prepared to calculate an error formula for either/both series tests and Taylor series

Quiz #10 is due tonight on Tuesday's material (finding a Taylor series)
Quiz #11 is on the material from tonight. It's not due until after the exam, but if you want feedback before taking the test, try to get submitted by Monday noon.