8/24/2023

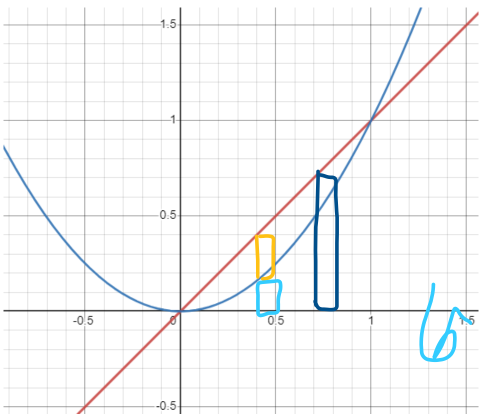
Volumes of Revolution: Disk/Washer (Slicing), Shells (2.2,2.3)

Disk/Washer (Slicing)

We are going to slice a solid of revolution into cross-sections that can be approximated as a cylinder.

Suppose that we want to find the volume of the solid of revolution of the region bounded by $y = x, y = x^2$, rotated around the x-axis.

For the Disk/Washer method, the axis of rotation and the variable in which the function is defined must agree.



How do we estimate the volume of the cylindrical disk or the annulus (disk with a hole)?

Volume of a cylinder: Area of the circle (base) times the height of the cylinder. The radius of the base is the height of the function (at that point), and the height of the cylinder is the width of the rectangle is Δx.

So $r = f(x_i)$

Volume = $\pi r^2(\Delta x) = \pi[f(x_i)]^2 \Delta x$

Volume of the annulus, is just the volume of the outer radius cylinder minus the volume of the hole

Volume = $\pi r_{outer}^2 \Delta x - \pi r_{inner}^2 \Delta x = \pi[f(x_i)]^2 \Delta x - \pi[g(x_i)]^2 \Delta x$

Add up the estimates to get the whole volume:

$$V_{total} = \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x$$

$$V_{solid} = \lim_{n\to\infty} \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x = \pi \int_a^b [f(x)]^2 dx$$

This is the disk method.
The washer method is

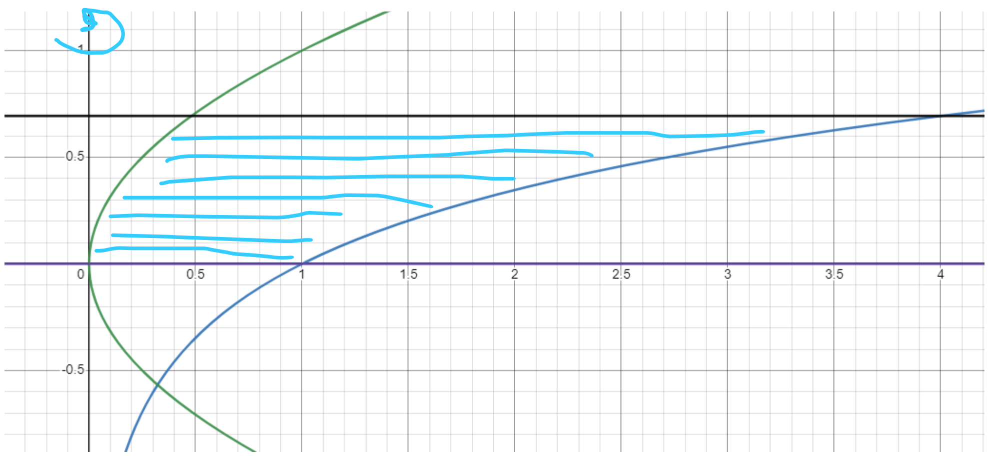
$$V_{solid} = \pi \int_a^b \{[f(x)]^2 - [g(x)]^2\}dx$$

The outer function is the one further from the axis of rotation. The inner function is the one close to the x-axis. You will need to use the washer method if any part of the inner function does not touch the x-axis.

$$V = \pi \int_0^1 (x)^2 - (x^2)^2 dx = \pi \int_0^1 x^2 - x^4 dx = \pi \left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$$

Rotating around the y-axis, if the functions are defined in terms of y.

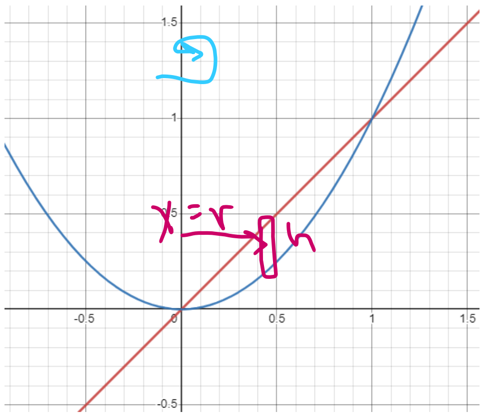
Find the volume of revolution of the solid formed by rotating the region bounded by $x = e^{2y}, x = y^2, y = 0, y = \ln 2$, around the y-axis.



$$V = \pi \int_0^{\ln 2} (e^{2y})^2 - (y^2)^2 dy = \pi \int_0^{\ln 2} e^{4y} - y^4 dy = \pi \left[\frac{1}{4}e^{4y} - \frac{1}{5}y^5\right]_0^{\ln 2} =$$

$$\pi\left[\frac{1}{4}e^{4\ln 2} - \frac{1}{5}\ln^5 2 - \frac{1}{4}e^0\right] = \pi\left[\frac{1}{4}(16) - \frac{1}{5}\ln^5 2 - \frac{1}{4}\right] = \pi\left[\frac{15}{4} - \frac{1}{5}\ln^5 2\right]$$

Shell method
Functions of x can be rotated around the y-axis; functions of y can be rotated around the x-axis.



Find the volume of the solid of revolution formed from the region bounded by $y = x, y = x^2$ rotated around the y-axis.

Sometimes this is called the method of cylindrical shells.

Volume=area of the rectangle (from unfolding the cylindrical shell) times the width of the original rectangle (shell) = (height of the function) times (the circumference of the shell) times Δx = $f(x_i)\ (2\pi\ times\ radius)\Delta x$= $f(x_i)2\pi x_i \Delta x$

Volume = $2\pi x_i f(x_i)\Delta x$

$$V_{total} = \sum_{i=1}^{n} 2\pi x_i f(x_i)\Delta x$$

$$V_{solid} = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i f(x_i)\Delta x = 2\pi \int_a^b x f(x)dx$$

Or

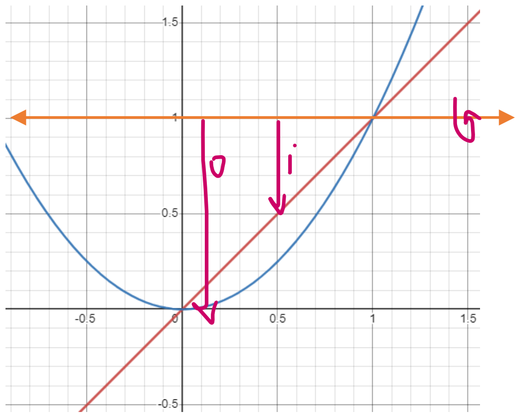
$$2\pi \int_a^b x[f(x) - g(x)]dx$$

$$V = 2\pi \int_0^1 x(x - x^2)dx = 2\pi \int_0^1 x^2 - x^3 dx = 2\pi \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = 2\pi[1/12] = \pi/6$$

What if we are rotating around an axis that is not the x-axis or the y-axis?

Rule 1: treat the line of rotation initially as if it's like the axis it is **parallel** to.
That means that a line like x=2 is similar to the y-axis. And a line line y=4 is similar to the x-axis.

Find the volume of the solid of revolution for the region bounded by $y = x, y = x^2$ rotated around the line $y = 1$.



This changes our orientation. The outer radius is now $y = x^2$ and the inner radius is now $y = x$. But since $y = 1$ is parallel to the x-axis, we use the washer method.
Outer radius = $axis\ of\ rotation - g(x) = 1 - x^2$
Inner radius = $axis\ of\ rotation - f(x) = 1 - x$

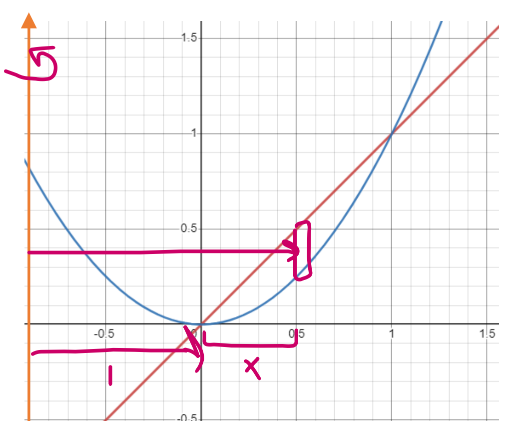
$$V = \pi \int_0^1 (1-x^2)^2 - (1-x)^2 dx = \pi \int_0^1 1 - 2x^2 + x^4 - (1 - 2x + x^2)dx =$$

$$\pi \int_0^1 1 - 2x^2 + x^4 - 1 + 2x - x^2 dx = \pi \int_0^1 2x - 3x^2 + x^4 dx = \pi \left[x^2 - x^3 + \frac{1}{5}x^5\right]_0^1 =$$

$$\pi \left[1 - 1 + \frac{1}{5}\right] = \frac{\pi}{5}$$

The radii change from just the function to either (the function minus the axis of rotation) or (the axis of rotation minus the function).

What if I wanted to rotate around the line $x = -1$?



This line is parallel to the y-axis. That means we use the shell method.

The height of the shells is unchanged by the new axis of rotation. What does change is the radius of the shell. Before: $r = x$, but now it's x minus the axis of rotation. $x - (-1) = x + 1$

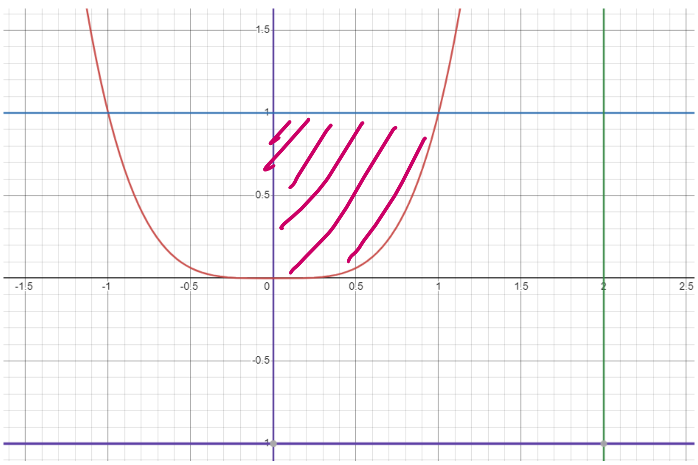
If I'm on the other side of the region from the y-axis, then the radius becomes axis of rotation minus x.

$$V = 2\pi \int_0^1 (x+1)(x-x^2)dx = 2\pi \int_0^1 x^2 - x^3 + x - x^2 dx = 2\pi \int_0^1 x - x^3 dx =$$

$$2\pi \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = 2\pi \left[\frac{1}{4}\right] = \frac{\pi}{2}$$

For the region bounded by $y = x^4, x = 0, y = 1$ find the volume of the solid of revolution rotated around:

a) $x - axis.$
b) $y-$axis
c) Line $y = -1$
d) Line $x = 2$



a) $V = \pi \int_0^1 1^2 - (x^4)^2 dx$
b) $V = 2\pi \int_0^1 x(1-x^4)dx$?=? $\pi \int_0^1 \left(\sqrt[4]{y}\right)^2 dy$
c) $V = \pi \int_0^1 [1-(-1)]^2 - [x^4 - (-1)]^2 dx$
d) $V = 2\pi \int_0^1 (2-x)(1-x^4)$

In general, try to the use the functions as given, rather than solve for a new variable, and select the method appropriate for the combination of variables and axes of rotation.