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Partial Fractions
Integrating Rational Functions
Overview of Integration Techniques: Identifying which techniques to use when

Integrating Rational Functions:

1) Long division: if the degree of the numerator is the same size or larger than the degree of the denominator, do long division before doing anything else.
2) Is this a log or inverse tangent function rule? (using substitution usually)
    a. Logarithmic integral: the numerator is one degree less than the denominator, and check to see if the numerator is a constant multiple of the derivative of the denominator.
    b. Inverse tangent rule: make the denominator into the sum of squares, is the numerator the derivative of the function being squared in the denominator.
3) Can the denominator be factored, and if so, then partial fractions may apply.

Long division example:

$$\int \frac{x^4+1}{x^2+1}dx \ or \ \int \frac{x^2-1}{x^2+1}dx$$

The numerator is a larger degree than the denominator. Do long division before proceeding.

Logarithmic integral:

$$\int \frac{x+1}{2x^2+4x-7}dx \ or \ \int \frac{x^2}{x^3+8}dx$$

Use u-sub with u = denominator, then end up with a logarithmic function after integrating

Inverse Tangent function:

$$\int \frac{1}{4+x^2}dx \ or \ \int \frac{x}{1+x^4}dx$$

Both of these are inverse tangent functions after integrating. In the second one, the denominator can be rewritten as $1+(x^2)^2$, making $u=x^2$, the derivative of this function is what you are comparing to the numerator.

Hint: If both terms contain x, or if there are three terms, you may need to complete the square in order to get the sum of squares to work out.

$$\int \frac{1}{x^2+4x+8}dx = \int \frac{1}{(x^2+4x+4)+4}dx = \int \frac{1}{(x+2)^2+4}dx$$

Partial fraction cases:
Proper fraction, and the denominator is factorable

$$\int \frac{x}{x^3-8}dx \ or \ \int \frac{x+3}{x^2-4x-12}dx = \int \frac{x+3}{(x-6)(x+2)}dx$$

The idea of partial fractions is to take a complex fraction and decompose it into simpler fractions.

$$\frac{1}{x+2}+\frac{3}{x-4}=\frac{1(x-4)}{(x+2)(x-4)}+\frac{3(x+2)}{(x+2)(x-4)}=\frac{x-4+3x+6}{(x+2)(x-4)}=\frac{4x+2}{(x+2)(x-4)}$$

Partial fractions tries to undo this combination process.

If I have the expression $\frac{4x+2}{(x+2)(x-4)}$ what are the numerators that I need to express this as two separate fractions with one factor in each denominator? $\frac{A}{x+2}+\frac{B}{x-4}$

How do we set up the partial fraction decomposition?

1) If the factors in the denominator are linear and not repeated, then the numerators are constants, and we have one factor in each term.

$$\frac{x^2+3x-7}{(x-2)(x+3)(x+1)}$$

Set-up for the partial fraction decomposition:

$$\frac{A}{x-2}+\frac{B}{x+3}+\frac{C}{x+1}$$

2) The factors are linear, but one or more is repeated. In this case, we have to have one fraction for every factor (no repetition), and one additional term with consecutive powers up to the most repetitions for any repeated factors.

$$\frac{x}{(x+1)^3(x-4)(x+5)}$$

Set up for the partial fraction decomposition:

$$\frac{A}{x-4}+\frac{B}{x+5}+\frac{C}{x+1}+\frac{D}{(x+1)^2}+\frac{E}{(x+1)^3}$$

Alternative set-up is: (but don't do it this way!!!)

$$\frac{A}{x-4}+\frac{B}{x+5}+\frac{Ex^2+Fx+D}{(x+1)^3}$$

3) What if one of the factors is an unfactorable quadratic? Like a sum of squares?
The decomposition will have a numerator one degree less than the factor in the denominator

$$\frac{x+1}{(x^2+1)(x-3)(x^2+x+2)}$$

Set-up for the partial fraction decomposition:

$$\frac{Ax+B}{x^2+1}+\frac{C}{x-3}+\frac{Dx+E}{x^2+x+2}$$

4) What if the quadratic factor is repeated?
Follow the same procedure for repeated linear factors, but with the linear numerator

$$\frac{2x+5}{(x^2+1)^2(x^2+4)}$$

Set up:

$$\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}+\frac{Ex+F}{x^2+4}$$

If you have a cubic factor or higher, you must find a way to factor.

Recall: Sum and difference of cubes formulas: $a^3-b^3=(a-b)(a^2+ab+b^2)$

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

Factoring by grouping if there are 4 terms.

Find one or more (rational roots) on your calculator and divide it out.

Solving.

$$\frac{4x+2}{(x+2)(x-4)}=\frac{A}{x+2}+\frac{B}{x-4}$$

Find a common denominator:

$$\frac{A(x-4)}{(x+2)(x-4)}+\frac{B(x+2)}{(x+2)(x-4)}=\frac{Ax-4A+Bx+2B}{(x+2)(x-4)}$$

$$Ax-4A+Bx+2B=4x+2$$

$$(Ax+Bx)+(-4A+2B)=4x+2$$

$$Ax+Bx=4x$$

Or

$$A+B=4$$

$$-4A+2B=2$$

$$A=4-B$$
$$-4(4-B)+2B=2$$

$$-16+4B+2B=2$$
$$-16+6B=2$$
$$6B=18$$
$$B=3$$

$$A=4-3=1$$

The decomposition is $\frac{1}{x+2}+\frac{3}{x-4}$

$$\int \frac{4x+2}{(x+2)(x-4)}dx = \int \frac{1}{x+2} + \frac{3}{x-4}dx = \ln|x+2| + 3\ln|x-4| + C$$

Another method of finding A and B without solving simultaneous equations:

$$\frac{A(x-4)}{(x+2)(x-4)} + \frac{B(x+2)}{(x+2)(x-4)} = \frac{A(x-4)+B(x+2)}{(x+2)(x-4)}$$

$$A(x-4) + B(x+2) = 4x + 2$$

Let x=4

$$A(0) + B(4+2) = 4(4) + 2$$
$$6B = 18$$
$$B = 3$$

Let x= -2

$$A(-2-4) + B(0) = 4(-2) + 2$$
$$-6A = -6$$
$$A = 1$$

If you have repeated factors, you may not be able to find all the constants with this trick, but it will reduce the problem to a smaller one. Then find the remaining constants after substituting what you can find.

You can also try picking a value of x like 0 to reduce the problem for the last coefficient.

If you have a quadratic factor, you may also not be able to make the factors 0. Use x=0, or x=1.

Example.

$$\int \frac{x}{x^3-8}dx = \int \frac{x}{(x-2)(x^2+2x+4)}dx$$

$$\frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4} = \frac{A(x^2+2x+4) + (Bx+C)(x-2)}{(x-2)(x^2+2x+4)}$$

$$A(x^2+2x+4) + (Bx+C)(x-2) = x$$

Approach 1:

$$Ax^2 + 2Ax + 4A + Bx^2 - 2Bx + Cx - 2C = x$$

$$A + B = 0 \ \ (x^2)$$
$$2A - 2B + C = 1 \ \ (x)$$
$$4A - 2C = 0 \quad (constants)$$

$$B = -A$$
$$4A = 2C \rightarrow C = 2A$$

$$2A - 2B + C = 1$$
$$2A - 2(-A) + 2(A) = 1$$
$$2A + 2A + 2A = 1$$
$$6A = 1$$
$$A = \frac{1}{6}$$
$$B = -\frac{1}{6}$$
$$C = \frac{1}{3}$$

Approach 2:

$$A(x^2 + 2x + 4) + (Bx + C)(x - 2) = x$$

Let x=2

$$A(4 + 4 + 4) + (Bx + C)(0) = 2$$
$$12A = 2$$
$$A = \frac{1}{6}$$

Try $x = 0$

$$A(0 + 0 + 4) + (B(0) + C)(0 - 2) = 0$$

$$4A - 2C = 0$$

Plug in A

$$2C = 4A$$
$$C = 2A = 2\left(\frac{1}{6}\right) = \frac{1}{3}$$

$$A(x^2 + 2x + 4) + (Bx + C)(x - 2) = x$$

Try x=1 and plug in A and C

$$\frac{1}{6}(1 + 2 + 4) + \left(B(1) + \frac{1}{3}\right)(1 - 2) = 1$$
$$\frac{1}{6}(7) - B - \frac{1}{3} = 1$$
$$-B + \frac{5}{6} = 1$$
$$-B = \frac{1}{6}$$
$$B = \frac{1}{6}$$

$$\int \frac{x}{x^3 - 8} dx = \int \frac{\frac{1}{6}}{x - 2} + \frac{\left(-\frac{1}{6}\right)x}{x^2 + 2x + 4} + \frac{\frac{1}{3}}{x^2 + 2x + 4} dx = \int \frac{\frac{1}{6}}{x - 2} + \frac{\left(-\frac{1}{6}\right)x + \frac{1}{3}}{x^2 + 2x + 4} dx$$

The first term is a log-rule

The second one might be a log rule, it might be a completing the square with a inverse tangent rule.

$$x^2 + 2x + 4 = (x + 1)^2 + 3$$