Math 2415/1285 Eigenvalues & Eigenfunctions Key

1. $y'' + \lambda y = 0$, $y(0)=0$, $y'(\pi)=0$

$r^2 + \lambda = 0 \Rightarrow r^2 = -\lambda \Rightarrow r = \pm\sqrt{-\lambda}$

Case 1: $\lambda = 0$ Then $r=0$ $\quad y = At + B$

$0 = y(0) = A(0) + B = B$ So $B = 0$

$y'(\pi) = A = 0$ So $A = 0$ and the solution is trivial.

Case 2: $\lambda = -\mu^2$ then $r = \pm\mu$

$y(t) = A\sinh(\mu t) + B\cosh(\mu t)$

$y(0) = A\sinh(0) + B\cosh(0) = B = 0$

$y'(t) = A\mu\cosh(\mu t)$ $\quad y'(\pi) = A\mu\cosh(\mu\pi) = 0$

this can only be zero if A is 0 since $\cosh(x)$ never $=0$.

Solution is trivial.

Case 3.

$\lambda = \mu^2$ then $r = \pm\mu i$

$y = A\cos(\mu t) + B\sin(\mu t)$

$y(0) = A + B(0) \Rightarrow A = 0$

$y'(t) = \mu B\cos(\mu t)$ $\quad y'(\pi) = \mu B\cos(\mu\pi) = 0$ $\quad \cos(\mu\pi) = 0$

if $\mu\pi = \frac{(2k+1)}{2}\pi \Rightarrow \mu = \frac{2k+1}{2}$ so $\lambda = \frac{(2k+1)^2}{4}$

and eigenfunction is $y(t) = B\sin\left(\left(\frac{2k+1}{2}\right)t\right)$

2. $y'' + \lambda y = 0$ $\quad y'(0) = 0$, $y'(L) = 0$

$r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda}$

Case 1: $\lambda = 0$ $\quad y(t) = At + B$ $\quad y'(0) = 0 = A$

$y'(L) = 0$ $\quad y(t) = B$ $\quad y'(0) = 0$

So eigenfunction is $y(t) =$ any constant.

Case 2: $\lambda = -\mu^2$ $\quad r = \pm\mu$

$y(t) = A\cosh\mu t + B\sinh\mu t$

2 cont'd

$y'(t) = \mu A\sinh(\mu t) + \mu B\cosh(\mu t)$ $y'(0) = \mu A(0) + \mu B(1) \Rightarrow B=0$

$y'(L) = \mu A\sinh(\mu L) = 0$ $\sinh(\mu L)$ only $=0$ if $\mu L = 0$ so $A=0$

trivial solution.

Case 3: $\lambda = \mu^2$ $r = \pm \mu i$

$y(t) = A\cos\mu t + B\sin\mu t$ $y'(t) = -\mu A\sin(\mu t) + \mu B\cos(\mu t)$

$y'(0) = -\mu A(0) + \mu B(1) \Rightarrow B=0$

$y'(L) = -\mu A\sin(\mu L)$ $\sin(\mu L) = 0$ when $\mu L = n\pi \Rightarrow \mu = \frac{n\pi}{L}$

$\lambda = \frac{n^2\pi^2}{L^2}$ and eigenfunction is $y(t) = A\cos\left(\frac{n\pi t}{L}\right)$

3. $y'' + 2\lambda y' + \lambda^2 y = 0$ $y(0)=0,\ y(L)=1$ $(r+\lambda)^2 = 0 \Rightarrow$
$r = -\lambda$ repeated

Case 1: $\lambda = 0 \Rightarrow r = 0$ $y(t) = At + B$

$y(0) = 0 = A(0) + B \Rightarrow B=0$ $y(L) = 1 = A(L)$ $A = \frac{1}{L}$

$y(t) = \frac{1}{L}t$

Case 2: $|\lambda| > 0$ $y(t) = Ae^{-\lambda t} + Bte^{-\lambda t}$

$y(0) = Ae^{-\lambda(0)} + B(0)e^{-\lambda(0)} \Rightarrow A = 0$

$y(L) = 1 = BLe^{-\lambda L} \Rightarrow \frac{1}{BL} = e^{-\lambda L} \Rightarrow -\ln(BL) = \lambda L \Rightarrow$

$\lambda = -\frac{1}{L}\ln(BL)$ this requires only that $BL > 0$

Solution is nontrivial

4. $y'' + \lambda y' + 8y = 0$ $y(0)=0,\ y(L)=0$

$r^2 + \lambda r + 8 = 0$ $\frac{-\lambda \pm \sqrt{\lambda^2 - 32}}{2} = r$

Case 1: $\lambda^2 - 32 = 0 \Rightarrow \lambda^2 = 32 = \pm 4\sqrt{2}$

$y(t) = A\cosh(4\sqrt{2}t) + B\sinh(4\sqrt{2}t)$

$y(0) = A(1) + B(0) \Rightarrow A = 0$

$y(L) = B\sinh(4\sqrt{2}L) = 0 \Rightarrow$ only 0 when $L=0$ or $B=0$

4 cont'd.

Case 2: $\lambda^2 - 32 > 0 = \mu^2 \Rightarrow \lambda^2 = \mu^2 + 32$

2 real roots

$r = \frac{-\lambda}{2} \pm \frac{1}{2}\mu \quad y(t) = Ae^{\left(\frac{-\lambda+\mu}{2}\right)t} + Be^{\left(\frac{-\lambda-\mu}{2}\right)t}$

$y(0)=0 \Rightarrow 0 = A(1) + B(1) = 0 \Rightarrow B = -A$

$y(L) = 0 \quad 0 = Ae^{\left(\frac{-\lambda+\mu}{2}\right)L} + Be^{\left(\frac{-\lambda-\mu}{2}\right)L}$

$0 = A\left[e^{\left(\frac{-\lambda+\mu}{2}\right)L} - e^{\left(\frac{-\lambda-\mu}{2}\right)L}\right] \Rightarrow 0 = e^{\left(\frac{-\lambda+\mu}{2}\right)L} - e^{\left(\frac{-\lambda-\mu}{2}\right)L}$

$e^{\left(\frac{-\lambda+\mu}{2}\right)L} = e^{\left(\frac{-\lambda-\mu}{2}\right)L} \Rightarrow \left(\frac{-\lambda+\mu}{\cancel{2}}\right)\cancel{L} = \left(\frac{-\lambda-\mu}{\cancel{2}}\right)\cancel{L} \Rightarrow \cancel{-\lambda}+\mu = \cancel{-\lambda}-\mu$

$\Rightarrow -\mu = \mu \Rightarrow \mu = 0$ this was case 1 so no new solutions

Case 3:

$\lambda^2 - 32 < 0 \Rightarrow -\mu^2 \Rightarrow r = \frac{-\lambda \pm \mu i}{2}$

$y(t) = Ae^{-\frac{\lambda}{2}t}\cos\left(\frac{\mu}{2}t\right) + Be^{-\frac{\lambda}{2}t}\sin\left(\frac{\mu}{2}t\right)$

$y(0) = A(1)(1) + B(1)(0) = 0 \Rightarrow A = 0$

$y(L) = 0 \Rightarrow Be^{-\frac{\lambda}{2}L}\sin\left(\frac{\mu}{2}L\right)$ exponential never zero, so

$\sin\left(\frac{\mu}{2}L\right) = 0$ when $\frac{\mu}{2}L = n\pi \Rightarrow \mu = \frac{2n\pi}{L} \Rightarrow \lambda = \pm\sqrt{32 - \frac{4n^2\pi^2}{L^2}}$

$\lambda^2 - 32 = -\mu^2 \Rightarrow \lambda^2 = 32 - \mu^2 \Rightarrow \lambda = \pm\sqrt{32-\mu^2}$ eigenvalue

$y(t) = Be^{\left[\frac{\pm\sqrt{32 - \frac{4n^2\pi^2}{L^2}}}{2}\right]t}\sin\left(\frac{2n\pi}{L}t\right)$

5. $y'' - \lambda y = 0 \quad y(0) = 0, \ y'(L) = 1$

$r^2 - \lambda = 0 \quad r = \pm\sqrt{\lambda}$

Case 1: $\lambda = 0 \Rightarrow r = 0 \quad y(t) = At + B$

$y(0) = 0 = A(0) + B \Rightarrow B = 0 \qquad y'(t) = A$

$y'(L) = 1 = A \Rightarrow A = 1 \qquad y(t) = t$ for $\lambda = 0$.

eigenfunction   eigenvalue

5 Cont'd. Case 2: ④

$\lambda > 0 \Rightarrow \lambda = \mu^2 \qquad r = \pm\mu$

$y(t) = A\cosh\mu t + B\sinh\mu t$

$y(0) = 0 = A(1) + B(0) \Rightarrow A = 0$

$y'(t) = \mu B\cosh\mu t \Rightarrow 1 = y'(L) = 1 = \mu B\cosh\mu L \Rightarrow B = 0$

Since $\cosh(\cancel{\mu}) \neq 0$ ∃ $\mu = 0$ is case 1.

Case 3:

$\lambda < 0 \Rightarrow \lambda = \mu^2 \qquad r = \pm\mu i$

$y(t) = A\cos(\mu t) + B\sin(\mu t) \qquad y(0) = A(1) + B(0) \Rightarrow A = 0$

$y'(t) = \mu B\cos(\mu t) \qquad y'(L) = 1 = \mu B\cos(\mu L)$

$\Rightarrow \frac{1}{\mu B} = \cos(\mu L)$ if $B = \frac{1}{\mu}$ Then $\cos(\mu L) = 1$ when

$\mu L = 2\pi n \Rightarrow \mu = \frac{2\pi n}{L}$ but other values of μ will also have other solutions so the set of eigenvalues is the set of real numbers. for some value of B.

Specifically $B = \frac{\sec(\mu L)}{\mu}$