Math 2568 # Inner Products Key

1. $u_1 \cdot u_2 = 1*0 + 0*1 + 1*0 = 0$ ✓

$u_2 \cdot u_3 = 0*1 + 1*0 + 0*(-1) = 0$ ✓ They are orthogonal

$u_1 \cdot u_3 = 1*1 + 0*0 + 1*(-1) = 0$ ✓

2. $u_1 \cdot u_2 = (-1)(-1) + 2(-3) + 1(0) + 0(5) + 0(1) = 1 - 6 = -5 \neq 0$

they are not orthogonal

3. let $u_2 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ $\quad u_1 \cdot u_2 = a - 3b + 5c + 4d = 0$

let $c = d = 0$

$a - 3b = 0$

$a = 3b$ $\quad$ let $b = 1, a = 3$ $\quad u_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Check $u_1 \cdot u_2 = 1(3) - 3(1) + 5(0) + 4(0) = 3 - 3 = 0$ ✓

let $u_3 = \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix}$

$u_1 \cdot u_3 = e - 3f + 5g + 4h = 0$

$u_2 \cdot u_3 = 3e + f + 0 + 0 = 0$

$\text{rref} \begin{bmatrix} 1 & -3 & 5 & 4 \\ 3 & 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1/2 & 2/5 \\ 0 & 1 & -3/2 & -6/5 \end{bmatrix}$

$e = -\frac{1}{2}g - \frac{2}{5}h$

$f = \frac{3}{2}g + \cancel{\frac{6}{5}}h$ $\quad$ let $h = 0$

$e = -\frac{1}{2}g$

$f = \frac{3}{2}g$ $\quad$ if $g = 2 \Rightarrow e = -1, f = 3$ $\quad u_3 = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 0 \end{bmatrix}$

$u_1 \cdot u_3 = 1(-1) - 3(3) + 5(2) + 4(0) = -1 - 9 + 10 + 0 = 0$ ✓

$u_2 \cdot u_3 = 3(-1) + 1(3) + 2(0) + 0(0) = -3 + 3 = 0$ ✓

$\left\{ \begin{bmatrix} 1 \\ -3 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 2 \\ 0 \end{bmatrix} \right\}$

4. $\langle \cos mx, \cos nx \rangle = \int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx =$

$\frac{1}{2}\int_{-\pi}^{\pi} \cos[(m-n)x] + \cos[(m+n)x] \, dx = \frac{1}{2}\left[\frac{\sin[(m-n)x]}{m-n} + \frac{\sin[(m+n)x]}{m+n}\right]_{-\pi}^{\pi}$

$= 0$ since integer multiples of π are always $= 0$

$\therefore$ these functions are orthogonal.

5. $\langle 1, x \rangle = \int_{-1}^{1} 1 \cdot x \, dx = \int_{-1}^{1} x \, dx = \frac{1}{2}x^2 \Big|_{-1}^{1} = \frac{1}{2}[(1)^2 - (-1)^2] = 0$ ✓

$\langle 1, 1-3x^2 \rangle = \int_{-1}^{1} (1)(1-3x^2) \, dx = \int_{-1}^{1} 1-3x^2 \, dx = x - x^3 \Big|_{-1}^{1} = 1 - 1^3 - (-1) - (-1)^3$
$= 0$ ✓

$\langle 1, -3x+5x^3 \rangle = \int_{-1}^{1} (1)(-3x+5x^3) \, dx = \int_{-1}^{1} -3x+5x^3 \, dx = 0$ odd function

$\langle x, 1-3x^2 \rangle = \int_{-1}^{1} x(1-3x^2) \, dx = \int_{-1}^{1} x - 3x^3 \, dx = 0$ odd function

$\langle x, -3x+5x^3 \rangle = \int_{-1}^{1} x(-3x+5x^3) \, dx = \int_{-1}^{1} -3x^2 + 5x^4 \, dx = -x^3 + x^5 \Big|_{-1}^{1} =$

$-(1)^3 + (1)^5 + (-1)^3 - (-1)^5 = 0$ ✓

$\langle 1-3x^2, -3x+5x^3 \rangle = \int_{-1}^{1} (1-3x^2)(-3x+5x^3) \, dx = \int_{-1}^{1} -3x + 5x^3 + 9x^3 - 15x^5 \, dx$

$= 0$ odd function

all these polynomials are orthogonal to each other

6. $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$

$\langle 1, p(x) \rangle = \int_{-1}^{1} (1)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4) \, dx =$

$a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \frac{a_4}{5}x^5 \Big|_{-1}^{1} =$

$a_0(1) + \cancel{\frac{a_1}{2}(1)} + \frac{a_2}{3}(1) + \cancel{\frac{a_3}{4}(1)} + \frac{a_4}{5}(1) + a_0(1) - \cancel{\frac{a_1}{2}(1)} + \frac{a_2}{3}(1) - \cancel{\frac{a_3}{4}(1)} + \frac{a_4}{5}$

6 Cont'd

$$\Rightarrow \quad 2a_0 + \frac{2a_2}{3} + \frac{2a_4}{5} = 0$$

$$\langle x, P(x)\rangle = \int_{-1}^{1} (x)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4)\,dx =$$

$$\int_{-1}^{1} a_0x + a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5\,dx =$$

$$\cancel{\frac{a_0}{2}x^2} + \frac{a_1}{3}x^3 + \cancel{\frac{a_2}{4}x^4} + \frac{a_3}{5}x^5 + \cancel{\frac{a_4}{6}x^6}\Big|_{-1}^{1} = \frac{2a_1}{3} + \frac{2a_3}{5} = 0$$

$$\langle 1-3x^2, P(x)\rangle = \int_{-1}^{1} (1-3x^2)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4)\,dx =$$

$$\int_{-1}^{1} a_0 + \cancel{a_1x} + a_2x^2 + \cancel{a_3x^3} + a_4x^4 - 3a_0x^2 - \cancel{3a_1x^3} - 3a_2x^4 - \cancel{3a_3x^5} - 3a_4x^6\,dx$$

$$= a_0x + \frac{a_2}{3}x^3 + \frac{a_4}{5}x^4 - \frac{3a_0}{3}x^3 - \frac{3a_2}{5}x^5 - \frac{3a_4}{7}x^7\Big|_{-1}^{1}$$

$$\cancel{a_0(1)} \cancel{+a_0(+1)} + \frac{a_2}{3} + \frac{a_2}{3} + \frac{a_4}{5} + \frac{a_4}{5} - \cancel{a_0} - \cancel{a_0} - \frac{3a_2}{5} - \frac{3a_2}{5} - \frac{3a_4}{7} - \frac{3a_4}{7}$$

$$= \left(\frac{2a_2}{3}\cdot\frac{5}{5} - \frac{6a_2}{5}\cdot\frac{3}{3}\right) + \left(\frac{2a_4}{5}\cdot\frac{7}{7} - \frac{6a_4}{7}\cdot\frac{5}{5}\right) = a_2\left(\frac{10-18}{15}\right) + a_4\left(\frac{14-30}{35}\right) =$$

$$-\frac{8a_2}{15} - \frac{16a_4}{35} = 0$$

$$\langle -3x+5x^3, p(x)\rangle = \int_{-1}^{1} (-3x+5x^3)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4)\,dx =$$

$$\int_{-1}^{1} \cancel{-3a_0x} - 3a_1x^2 - \cancel{3a_2x^3} - 3a_3x^4 - \cancel{3a_4x^5} + \cancel{5a_0x^3} + 5a_1x^4 + \cancel{5a_2x^5} +$$

$$5a_3x^6 + \cancel{5a_4x^7}\,dx =$$

$$\int_{-1}^{1} -3a_1x^2 - 3a_3x^4 + 5a_1x^4 + 5a_3x^6\,dx =$$

$$-\frac{3a_1}{3}x^3 - \frac{3a_3}{5}x^5 + \frac{5a_1}{5}x^5 + \frac{5a_3}{7}x^7\Big|_{-1}^{1}$$

$$\cancel{-2a_1} - \frac{6a_3}{5} + \cancel{2a_1} + \frac{10a_3}{7} = \frac{-42a_3 + 50a_3}{35} = \frac{8a_3}{35} = 0$$

$a_3 = 0$

$\Rightarrow a_1 = 0$

$$2a_0 + \frac{2}{3}a_3 + \frac{2}{5}a_4 = 0$$
$$-\frac{8}{15}a_2 - \frac{16}{35}a_4$$

$$\begin{array}{ccc} a_0 & a_2 & a_4 \end{array}$$
$$\begin{bmatrix} 2 & 2/3 & \frac{2}{5} \\ 0 & -\frac{8}{15} & -\frac{16}{35} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -3/35 \\ 0 & 1 & 6/7 \end{bmatrix}$$

6 cont'd

$a_0 = 3/35\, a_4$

$a_2 = -\frac{6}{7} a_4$ let $a_4 = 35$ $\Rightarrow$ $a_0 = 3$, $a_2 = -\frac{6}{\cancel{7}}(\cancel{35}^{5}) = -6(5) = -30$

$p(x) = 3 - 30x^2 + 35x^4$

you can check by redoing the inner product calculations in your calculator that this function is orthogonal to the other functions.

7. $\langle xe^{ix^2}, xe^{-ix^2} \rangle = \int_{-\infty}^{\infty} xe^{-ix^2} \cdot xe^{-ix^2}\, dx = \int_{-\infty}^{\infty} x^2 e^{-2ix^2}\, dx =$

$\overline{xe^{ix^2}} = xe^{-ix^2}$

$u = x \quad dv = xe^{-2ix^2}$

$du = dx \quad v = -\frac{1}{4i} e^{-2ix^2}$

$$\lim_{b\to\infty} \int_{-b}^{0} x^2 e^{-2ix^2}\, dx + \lim_{b\to\infty} \int_{0}^{b} x^2 e^{-2ix^2}\, dx$$

$$\lim_{b\to\infty} \left[\cancel{\frac{-x}{4i} e^{-2ix^2}}\Big|_{-b}^{0} + \int_{-b}^{0} \frac{1}{4i} e^{-2ix^2}\, dx + \cancel{\frac{-x}{4i} e^{-2ix^2}}\Big|_{0}^{b} + \int_{0}^{b} \frac{1}{4i} e^{-2ix^2}\, dx \right]$$

$$= \lim_{b\to\infty} \frac{1}{4i} \left[\int_{-b}^{0} e^{-2ix^2}\, dx + \int_{0}^{b} e^{-2ix^2}\, dx \right] =$$

$$\frac{1}{4i} \int_{-\infty}^{\infty} e^{-2ix^2}\, dx = \frac{1}{4i} \left[\frac{1}{2} - \frac{i}{2} \sqrt{\pi} \right] \neq 0$$

you can get this from Wolfram Alpha since the function isn't integrable by traditional techniques

these functions are not orthogonal under this inner product.