Line Integrals

In order to do line integrals in the usual way, we first need to know how to parameterize curves. We are going to start with some basic examples that will be useful in doing the line integrals to follow.

Example 1. Straight lines or line segments.
The most basic example involves connecting two points. The simplest way to do this is with a straight line. In many cases, it won't matter how we get from point *A* to point B, so the simplest route is the best. Consider two points in 3-space:*A* (1,5,9), and B(-2,7,10). You can use the same method in 2D.

First, calculate the vector that connects the two points.

-2-1=-3 7-5=2 10-9=1

$\vec{v} = \langle -3, 2, 1 \rangle$

Now, describe the path as

$\vec{r}(t) = \langle -3, 2, 1 \rangle t + A = \langle -3t+1, 2t+5, t+9 \rangle = (-3t+1)\vec{i} + (2t+5)\vec{j} + (t+9)\vec{k}$

This is the same thing as saying $x = -3t+1, y = 2t+5, z = t+9$. When we do line integrals, we will use these forms interchangeably. It's also important to note that you travel from point *A* to point *B* on the interval $0 \le t \le 1$.

If you have to connect through several points, repeat this procedure. When we do line integrals, we will be integrating each segment of the path separately, so, you can integrate several times over the interval [0,1]. However, you will sometimes see the second segment being done on the interval $1 \le t \le 2$, and so forth. To get the equations in this form, instead of multiplying the vector by *t*, replace *t* with by *t-1* for $1 \le t \le 2$, *t-2* for $2 \le t \le 3$, and so forth.

Example 2. Curves like circles and ellipses.
For the path along the circle $(x-h)^2 + (y-k)^2 = a^2$, where (*h*,*k*) is the center of the rotation, and *a* is the radius, use the parameterization $\vec{r}(t) = (a\cos t + h)\vec{i} + (a\sin t + k)\vec{j}$, assuming that you start in the direction of the positive *x*-axis. If you start elsewhere on the circle, you can shift *t* by multiples of π/2. You can also switch the sine and cosine if you want to go around the circle in the clockwise direction instead of the standard counterclockwise direction.

For the path along the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ use the parameterization $\vec{r}(t)=a\cos t\vec{i}+b\sin t\vec{j}$. You can shift the center just as we did for the circle. In 3-space, you can turn this into a helix, or an elliptical helix by adding a *k*-component with some multiple of *t*. In both cases the path will be counterclockwise. Choose your t-values to specify the correct starting location.

Example 3. Other curves.
For all other curves, write an equation in two or three variables. For two variables, choose *x* to be *t* and that will give you the expression for *y*. For three variables, you will need a minimum of two equations to get a curve instead of surface. You can solve for a two variable equation and then repeat the above process. Use your parameterizations for the first two variables to find the third in the system.

Line Integrals.
Let us now return to the topic of line integrals.

There are several different notations for line integrals, so we need to consider these first. One common notation is $\int_C f(x,y)ds \quad or \quad \int_C f(x,y,z)ds$. If the curve is closed (it starts and stops in the same place, then it may look like this $\oint_C f(x,y,z)ds$. The circle in the notation just emphasizes that the curve is closed. Work is a common application for line integrals: $\int_C \vec{F}\cdot d\vec{r}$. And they may also look like $\int_C M(x,y)dx+N(x,y)dy \quad or \quad \int_C M(x,y,z)dx+N(x,y,z)dy+P(x,y,z)dz$. In all these cases, C is some curve in 2- or 3-space. All these forms look a bit different, but the last two at least are, in the end, equivalent forms. Let's do each one in turn. The first form is used for calculating arc length if f(x,y,z)=1, or mass of a wire. The second two are typically involved in calculating work (or energy) in a vector field.

Example 4. Evaluate the line integral $\int_C 2xyzds$ on the curve *C*: $\vec{r}(t)=12t\vec{i}+5t\vec{j}+84t\vec{k}$, $0\le t\le 1$.

In the function $2xyz$, replace *x=12t, y=5t* and *z=84t* as given by the parameterization of the curve. This becomes $2xyz=2(12t)(5t)(84t)=10080t^3$. The ds part of the equation should be replaced with the arc length of the curve: $ds=\left\|\vec{r}'(t)\right\|dt$.

$\vec{r}'(t) = 12\vec{i} + 5\vec{j} + 84\vec{k}$

$\left\|\vec{r}'(t)\right\| = \sqrt{12^2 + 5^2 + 84^2} = \sqrt{7225} = 85$

We can now write our line integral as $\int_0^1 10080t^3(85)dt = 856,800\int_0^1 t^3 dt = 214,200$.

The limits of the integral come from the limits for *t* along the path.

Example 5. Evaluate $\oint_C F(x,y,z)ds$ along the closed curve $C:\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + 4\vec{k}$, on the interval $0 \le t \le 2\pi$ for $F(x,y,z) = xy$.

As before, use the parameterization to write the function in terms of *t*. $F(x,y,z) = xy = \cos t \sin t$.

$\vec{r}'(t) = -\sin t\vec{i} + \cos t\vec{j}$

$\left\|\vec{r}'(t)\right\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$

So our line integral becomes: $\int_0^{2\pi} \cos t \sin t(1)dt = \frac{1}{2}\sin^2(t)\,|_0^{2\pi} = 0$.

Not all line integrals on closed curves will be zero, but they will all be zero if the function we are integrating is the gradient of a potential function, i.e. if the field is conservative. If the field is not conservative, all you can do is integrate and see.

Example 6. Calculate the work $\int_C \vec{F}\cdot d\vec{r}$ done moving a particle along the path $C:\vec{r}(t) = 2\sin t\vec{i} + 2\cos t\vec{j} + \frac{1}{2}t^2\vec{k}, 0 \le t \le \pi$ through the field $\vec{F}(x,y,z) = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$.

$dr = \vec{r}'(t)dt = \langle 2\cos t, -2\sin t, t\rangle dt$

Replace the variables in *F* as before.

$$\vec{F}(t) = \left\langle 4\sin^2 t, 4\cos^2 t, \frac{1}{4}t^4 \right\rangle$$

Dot the two vectors together and then integrate over the given interval.

$$\left\langle 4\sin^2 t, 4\cos^2 t, \frac{1}{4}t^4 \right\rangle \cdot \langle 2\cos t, -2\sin t, t \rangle dt = \left(8\sin^2 t\cos t - 8\cos^2 t\sin t + \frac{1}{4}t^5 \right) dt$$

$$\int_0^\pi 8\sin^2 t\cos t - 8\cos^2 t\sin t + \frac{1}{4}t^5 dt \approx 34.72$$

Example 7. Evaluate the line integral $\int_C (2x-y)dx + (x+3y)dy$ on the path from (0,0) to (0,-3) to (2,-3).

We are going to calculate the line integrals from (0,0) to (0,-3), and then (0,-3) to (2,-3) separately and then add the results.

On the first section of the path, the vector connecting the points is <0,-3>. So the parameterized path is $\vec{r}_1(t) = -3t\vec{j}, 0 \le t \le 1$. The variable x is always zero, as therefore, is dx. The integral reduces to:

$$\int_0^1 3ydy = \int_0^1 3(-3t)(-3dt) = \int_0^1 27tdt = \frac{27}{2}$$

On the second section of the path, the vector connecting the points is <2,0>. So the parameterized path is $\vec{r}_2(t) = 2t\vec{i} - 3\vec{j}, 0 \le t \le 1$. The y variable derivative is always zero, so the second term disappears.

The integral reduces to:

$$\int_C (2x-y)dx + (x+3y)dy = \int_0^1 (2)(2t-(-3))(2dt) = 4\int_0^1 2t+3dt = 4\left[t^2+3t\right]_0^1 = 4[1+3] = 16$$

Adding the two together, we find the integral evaluates to $\frac{27}{2} + 16 = \frac{59}{2}$.

Practice Problems. Evaluate the line integrals in each of the problems below.

1. $\int_C 3(x-y)ds, C: \vec{r}(t) = t\vec{i} + (2-t)\vec{j}, 0 \le t \le 2$

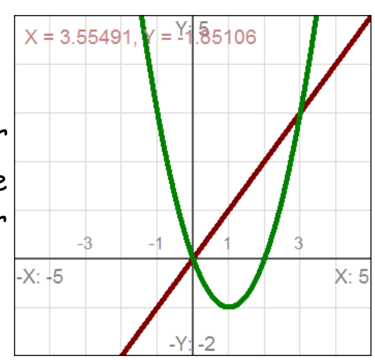
2. $\int_C (x^2 + y^2)ds$ along the line segment from (0,0) to (2,4). Is the value the same if the path is an ellipse, with center at (0,4) and a=4,b=2. [Hint: draw the path and check your equation to make sure they line up at the correct points.]
3. $\int_C \rho(x,y,z)ds, \rho(x,y,z) = kz, \vec{r}(t) = t^2\vec{i} + 2t\vec{j} + t\vec{k}, 1 \le t \le 3$
4. $\int_C \vec{F} \cdot d\vec{r}, F(x,y) = x\vec{i} + y\vec{j}, C : \vec{r}(t) = t\vec{i} + t\vec{j}, 0 \le t \le 1$
5. $\int_C \vec{F} \cdot d\vec{r}, F(x,y,z) = x^2z\vec{i} + 6y\vec{j} + yz^2\vec{k}, C : \vec{r}(t) = t\vec{i} + t^2\vec{j} + \ln t\vec{k}, 1 \le t \le 3$
6. $\int_C \vec{F} \cdot d\vec{r}, F(x,y,z) = yz\vec{i} + xz\vec{j} + xy\vec{k}, C : line\ from\ (0,0,0)\ to\ (5,3,2)$
7. $\int_C (3y - x)dx + y^2dy, C : \vec{r}(t) = 2t\vec{i} + 10t\vec{j}, 0 \le t \le 1$
8. $\int_C \vec{F} \cdot d\vec{r}, F(x,y,z) = 2xy\vec{i} + y^2z\vec{j} - x^2zy\vec{k}, C : line\ from\ (0,0,0)\ to\ (2,0,1),\ then\ to\ (2,1,4)$
9. $\int_C \left(x^2 + y^2\right)dx + 2xydy, \vec{r_1}(t) = t^3\vec{i} + t^2\vec{j}, 0 \le t \le 2; \vec{r_2}(t) = 2\cos t\vec{i} + 2\sin t\vec{j}, 0 \le t \le \frac{\pi}{2}$
10. Determine if any of the fields in 4-9 represent either conservative fields or closed curves, or both.

Green's Theorem.

Green's Theorem is a way of evaluating line integrals along the boundary of some region by differentiating the functions (to make them simpler), and then completing a double integral over the region. It converts a linear path to analyzing area. This may seem more complicated at first, but because the functions are generally simpler, and we can change coordinate systems.

The formula we'll be using for these problems is $\int_C M(x,y)dx + N(x,y)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$

Example 8. Use Green's Theorem to evaluate the integral $\int_C (y - x)dx + (2x - y)dy$ on the path along the boundary of the region lying between the graphs of $y = x, y = x^2 - 2x$.



Let's look at the graph to see what this path looks like.

In this example $M(x,y) = y - x$ and $N(x,y) = 2x - y$. Find their partial derivatives. If the field is conservative, you'll get the same answer for both derivatives, and so, as expected, your answer will be zero on a closed curve.

$$\frac{\partial N}{\partial x} = 2, \frac{\partial M}{\partial y} = 1$$

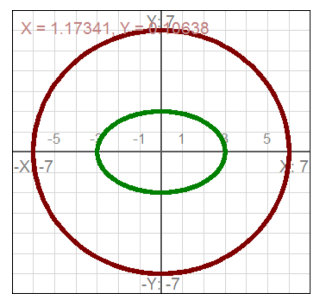
$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = 2 - (1) = 1 \Rightarrow \iint_R 1dA$$

The limits in y are our two functions, and in x, it's the intersection points at 0 and 3.

$$\int_0^3 \int_{x^2-2x}^{x} dydx = \int_0^3 [y]_{x^2-2x}^{x} dx = \int_0^3 x - (x^2 - 2x)dx = \int_0^3 3x - x^2 dx = \frac{3}{2}x^2 - \frac{1}{3}x^3 |_0^3 = \frac{9}{2}$$

Check this answer by parameterizing the curve and see if you get the same answer. (This will be one of the practice problems.)

Example 9. Use Green's Theorem to evaluate the integral $\int_C (e^{-x^2/2} - y)dx + (e^{-y^2/2} + x)dy$ on the path along the boundary of the region lying between the graphs of the circle $x = 6\cos\theta, y = 6\sin\theta$ *and ellipse* $x = 3\cos\theta, y = 2\sin\theta$.



Let's consider the graph.

To travel along the boundary of this graph, we will have to connect the two curves. Since we will travel from the ellipse to the circle, as long as travel back to the ellipse (to close the curve) along the same path, then this segment of the path will not affect the value of the integral. The nice thing with Green's Theorem is you don't have to find the exact curve in order to do the integral.

We really wouldn't want to have to integrate this by line integrals; integrating exponentials with trig functions in them is no fun. Using Green's Theorem, we need to find the derivatives of our functions.

$$N(x,y) = e^{-y^2/2} + x \Rightarrow \frac{\partial N}{\partial x} = 1; M(x,y) = e^{-x^2/2} - y \Rightarrow \frac{\partial M}{\partial y} = -1$$

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = 1 - (-1) = 2 \Rightarrow \iint_R 2dA = 2\iint_R dA$$

Integrating this in polar won't be great fun. The outer boundary is just r=6. But the inner ellipse becomes: $r = \dfrac{6}{\sqrt{4 + 5\sin^2\theta}}$. This won't be fun to integrate at all, but there is a way out. The double integral over a region like this is just the area of the region. Rather than integrating this one, you can use the formulas. The area of the circle is $A = \pi r^2 = 36\pi$. The area of the ellipse is $A = ab\pi = (3)(2)\pi = 6\pi$. Subtract to get the area of the region. The area is then $A = 36\pi - 6\pi = 30\pi$. Multiply by 2 to get the value of the line integral. [Note: this method only works if the function you are integrating is a constant. If any variables remain, you will have to do the awful integration, though you may be able to do it numerically.]

Alternative forms of Green's Theorem exist but are pretty uncommon, except in particular applications. The theorem says that:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_R \left(curl \vec{F} \right) \cdot \vec{k} dA = \iint_R \left(\nabla \times \vec{F} \right) \cdot \vec{k} dA.$$

Where $\vec{k} = <0,0,1>$, which eliminates any terms of the curl in the other two dimensions. Extending this version to three dimensions is called Stokes' Theorem, which will be looked at below.

The theorem can also be written using the normal to the surface:

$$\int_C \vec{F} \cdot \vec{N} ds = \iint_R div \vec{F} dA = \iint_R \left(\vec{\nabla} \cdot \vec{F} \right) dA = \int_R \int \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA.$$

The three dimensional analog of this is called the Divergence Theorem.

Common applications of line integrals include work through a force field, and area, using the formula A=½ $\int_C y dx + x dy$ using the boundary of the region in question as the path, which is easily shown using Green's Theorem.

Practice Problems.

11. For any problems in the first set of practice problems that were closed curves, redo them using Green's Theorem.
12. Do Example 8 without Green's Theorem and verify the result.
13. Evaluate $\int_C 2xy dx + (x+y) dy$ boundary of region between $y = 0, y = 1 - x^2$
14. Evaluate $\int_C \cos y dx + (xy + x \sin y) dy$ boundary of region between $y = x, y = \sqrt{x}$
15. Use Green's Theorem to prove that $\int_C f(x) dx + g(y) dy = 0$.

Stokes' Theorem.

Stokes' Theorem $\int_C \vec{F} \cdot d\vec{r} = \int_R \int (\nabla \times \vec{F}) \cdot \nabla G dA$ can be seen as a way of calculating line integrals by means of surface integrals. As we mentioned above, this is the more general version of Green's Theorem. It applies to surfaces other than the flat xy-plane (as

Green's Theorem does since the k-vector is normal (perpendicular) to it, and it applies to situations where three variables are used rather than just the two available to us in Green's Theorem.

Converting a line integral into a surface integral introduces one possible area of variation: many surfaces can have a single curve as its boundary. Consider a circle of radius 3 in the xy-plane. We can use the plane itself as the surface, or we could use a hemisphere of radius 3 as the surface or the paraboloid $z = 9 - x^2 - y^2$, which will also intersect in the plane in a circle of radius 3, or any of a number of other possible functions. Indeed, there is an infinite number of such functions. How do we know which to choose?

In the problems we will be doing, the surface will be provided for us in each problem, but in general, symmetry or other information available will generally suggest a good surface. When options are available, choose one that makes the calculations as simple as possible.

However, in practice, Stokes' Theorem is sometimes applied in the reverse direction, where a surface integral is converted into a line integral, in which case, the surface is specified as part of the problem. For such a problem, we would need to know the field F in addition to the curl(F) since calculating F from the curl alone challenging even in the best of cases.

Example 10. Apply Stokes' Theorem to calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$ using the field $\vec{F}(x,y,z) = z^2\hat{\imath} + 2x\hat{\jmath} + y^2\hat{k}$ on surface $S: z = 9 - x^2 - y^2, z \geq 0$.

To begin this problem, we will need to calculate the curl of the field.

$$\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 & 2x & y^2 \end{vmatrix} = (2y - 0)\hat{\imath} - (0 - 2z)\hat{\jmath} + (2 - 0)\hat{k} = 2y\hat{\imath} + 2z\hat{\jmath} + 2\hat{k}$$

For more information on finding G(x,y,z) see the surface integrals handout and oriented surfaces. Here, the orientation matters only so that the sign is correct.

We find $G(x,y,z) = z - 9 + x^2 + y^2$ by pointing all the terms in the surface on one side of an equation set equal to zero, and with the leading z-term positive. From here calculate the gradient (and thus the normal to the surface) of G.

$$\nabla G = 2x\hat{\imath} + 2y\hat{\jmath} + 1\hat{k}$$

In order to integrate over the surface, we need to reduce both these vectors, and thus any resulting dot product to just two vectors. Replace any z's in the equation with the equation of the surface solved for z.

$$\nabla \times F = 2y\hat{\imath} + 2z\hat{\jmath} + 2\hat{k} = 2y\hat{\imath} + (18 - 2x^2 - 2y^2)\hat{\jmath} + 2\hat{k}$$

Now, dot these two vectors together:

$$\left[2y\hat{\imath} + (18 - 2x^2 - 2y^2)\hat{\jmath} + 2\hat{k}\right] \cdot \left[2x\hat{\imath} + 2y\hat{\jmath} + 1\hat{k}\right] = 4xy + 36y - 4y(x^2 + y^2) + 2$$

This is the function that will have to be integrated over the 2-dimensional region in the plane. The surface given intersects with z=0 in the circle of radius 3. Given that, it's probably best to convert this problem entirely to polar.

Thus,

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \int_0^3 (4r^2 \sin\theta \cos\theta + 36r \sin\theta - 4r^3 \sin\theta + 2) r dr d\theta$$

Integrating then:

$$\int_0^{2\pi} \int_0^3 (4r^3 \sin\theta \cos\theta + 36r^2 \sin\theta - 4r^4 \sin\theta + 2r) dr d\theta$$

$$= \int_0^{2\pi} 81 \sin\theta \cos\theta + 324 \sin\theta - \frac{972}{5} \sin\theta + 9 d\theta =$$

$$\int_0^{2\pi} 81 \sin\theta \cos\theta + \frac{648}{5} \sin\theta + 9 d\theta = \frac{81}{2} sin^2\theta - \frac{648}{5} \cos\theta + 9\theta \Big|_0^{2\pi} = 18\pi$$

Practice Problems. Use Stokes' Theorem to find the value of the line integral $\int_C \vec{F} \cdot d\vec{r}$ using the given surface.

16. $\vec{F}(x,y,z) = x^2\hat{\imath} + z^2\hat{\jmath} - xyz\hat{k}, S: z = \sqrt{4 - x^2 - y^2}$
17. $\vec{F}(x,y,z) = yz\hat{\imath} + (2 - 3y)\hat{\jmath} + (x^2 + y^2)\hat{k}, S: first\ octant\ portion\ of\ x^2 + z^2 = 16\ over\ x^2 + y^2 = 16$
18. $\vec{F}(x,y,z) = 2y\hat{\imath} + 3z\hat{\jmath} - x\hat{k}, S: z = 9 - 2x - 3y\ over\ r = 2\sin 2\theta\ \ in\ the\ first\ octant$
19. $\vec{F}(x,y,z) = z^2\hat{\imath} + x^2\hat{\jmath} + y^2\hat{k}, S: z = y^2, 0 \le x \le a, 0 \le y \le a$