

## Integration by Substitution

Integration by substitution is a technique that allows us to find antiderivatives of functions derived via the chain rule.

Recall, the chain rule is used for taking derivatives of complicated functions created by composing two functions:  $f(x)$  and  $g(x)$  becomes  $h(x) = f(g(x))$ .

For instance, if  $f(x) = e^x$  and  $g(x) = x^2 + 2$ , then  $h(x) = e^{x^2+2}$ . Following the chain rule,  $h'(x) = f'(g(x)) \cdot g'(x) = e^{x^2+2} \cdot 2x$ .

Integration by substitution can get us from  $\int -2xe^{x^2+2} dx$  back to  $e^{x^2+2} + C$  without explicitly knowing the result in advance.

There are two common forms of functions that can be solved by substitution. They will be

1.  $\int f(x) \cdot f'(x) dx$  or
2.  $\int f(g(x))g'(x) dx$

\*The first case is really a special case of the second, but it shows up so often, we will treat it separately.

In case #1, we have a simple function times its own derivative, for example  $\int \frac{\ln x}{x} dx$ . In case #2, we have a complicated composite function times the derivative of the “inside”, for example  $\int 3x^2 e^{x^3} dx$  or our example from above  $\int -2xe^{x^2+2} dx$ .

### Case #1.

If your problem is of the form  $\int f(x) \cdot f'(x) dx$ , then let  $f(x) = u$  which will be our substitution variable. If we take the derivative of this statement on both sides we get  $f'(x) dx = du$  using differential notation. Replace these things in the integral:  $f(x)$  with  $u$  and  $f'(x) dx$  with  $du$ . We get  $\int f(x) \cdot f'(x) dx = \int u du$ . Now just integrate according to the power rule:  $\int u du = \frac{1}{2} u^2 + C$  and then replace  $u$  with your function of  $x$  again:

$\frac{1}{2} u^2 + C = \frac{1}{2} [f(x)]^2 + C$ . You can check by applying the chain rule to see that you do get the expression originally in the integral.

**Example 1.**  $\int \frac{\ln x}{x} dx$

Let  $u = \ln x$  and then  $du = \frac{1}{x} dx$ .

This implies  $\int \frac{\ln x}{x} dx = \int u du = \frac{1}{2} u^2 + C = \frac{1}{2} \ln^2 x + C$ .

### Case #2.

Let's look at the slightly more complicated case of  $\int f(g(x))g'(x)dx$ . Here we want  $g(x) = u$  and if we take the derivative on both sides as before we get  $g'(x)dx = du$ . What we get then is  $\int f(g(x))g'(x)dx = \int f(u)du$ . But at least now we have a simpler function to integrate and no product to worry about. Integrate with the u-function, and then be sure to replace everything back to your x-function to finish.

**Example 2.**  $\int 3x^2 e^{x^3} dx$

In an exponential function, the "inside" g function is the exponent of the exponential. So let  $u = x^3$ , taking the derivative we get  $3x^2 dx = du$ . So  $\int 3x^2 e^{x^3} dx = \int e^{x^3} \cdot 3x^2 dx = \int e^u du$ . This is easy to integrate from here.  $\int e^u du = e^u + C = e^{x^3} + C$  once we put everything back.

### Tips for choosing u.

Sometimes only experience tells you what to choose, but there are some usual suspects.

1. If you have either a log function or an inverse trig function, it must be u since we cannot obtain these functions by taking derivatives (i.e. we can't integrate them).

a.  $\int \frac{2x \ln(x^2 + 1)}{x^2 + 1} dx$   $u = \ln(x^2 + 1)$

b.  $\int \frac{1}{(\log_4 p) p} dp$

$u = \log_4 p$  You'll want to look up how to do derivative for logs other than natural

logs. In this case:  $du = \frac{1}{p \ln 4} dp$

2. If you have an exponential function whose exponent is other than x (or whatever variable you are using), let the exponent be u. If you let the whole exponential be u, you will get another exponential when you take the derivative. So, if you have only one, using the whole thing won't help you.

a.  $\int 3x^2 e^{x^3} dx$   $u = x^3$

b.  $\int \frac{\ln x}{x} e^{\ln^2 x} dx$   $u = \ln^2 x$

3. If you have powers or roots, use the inside as u. Do not use the exponents, though.

$$\text{a. } \int (x^3 + 2x - 6)^4 (3x^2 + 2) dx \quad u = x^3 + 2x - 6$$

$$\text{b. } \int 2x\sqrt{1+x^2} dx \quad u = 1+x^2$$

4. If you have any denominators, if the numerator is one degree less than the denominator, there is a good chance (at least in the beginning here) to let the denominator be  $u$ . If there are exponents, though, see above. This can be tricky with trig functions.

$$\text{a. } \int \frac{2z}{1-z^2} dz \quad u = 1-z^2$$

$$\text{b. } \int \frac{2x-2}{x^2-2x+1} dx \quad u = x^2-2x+1$$

$$\text{c. } \int \frac{t^3}{t^4+1} dt \quad u = t^4+1$$

$$\text{d. } \int \frac{e^x}{e^x+3} dx \quad u = e^x+3$$

5. If you have a product of three or more terms, you can do a double substitution or recognize that 2 of the products are actually just coming from a double chain rule and do it in one step. To do that latter, look for the *most complicated product*, and take the largest inside function you can.

$$\text{a. } \int \frac{x \ln(x^2+1)}{x^2+1} dx$$

- i. You can choose  $u = x^2 + 1$  and since  $du = 2x dx$  this will reduce the problem to  $\frac{1}{2} \int \frac{\ln u}{u} du$ , but then you will have to do a second substitution in order to finish integrating. Choose a different variable, let  $w = \ln u, dw = \frac{1}{u} du$  and this will get us to a point where we can integrate finally:  $\frac{1}{2} \int w dw$ . (we will explain the constant below)
- ii. If, however, you choose  $u = \ln(x^2 + 1)$  to start with, you can get to this same integration point in one step. When you take the derivative, you will get both missing pieces from the chain rule since  $du = \frac{2x}{x^2+1} dx$ .

6. No substitutions can ever have your derivative (du) end up in the denominator. If it is, then your substitution is incorrect, or the integral can't be done by this method.

### What if your substitution and your integral don't exactly match up?

This happens most of the time, but as long as we are only off by a coefficient (a constant multiplier) we can correct for this by tweaking the constants. It's the variable part that must match up in all cases. As long as that is true, we can adjust the others.

**Example 3.**  $\int e^{3x} dx$

Here, we let  $u=3x$  and  $du=3dx$ . But we don't have a 3 you say? There are two ways to deal with this:

**Method #1:** Divide it to the other side:  $\frac{1}{3} du = dx$ . Now we can substitute.  $\int e^{3x} dx = \int e^u \cdot \frac{1}{3} du$ .

We can rearrange to move the constant out of the way, and integrate.

$\frac{1}{3} \int e^u du = \frac{1}{3} e^u + C = \frac{1}{3} e^{3x} + C$ . And of course, you can always check that when you take the derivative, the factor 3 from the chain rule will cancel out and we'll get the function we started with.

**Method #2:** The other alternative is to multiply the entire integral by one in the form here of  $\frac{1}{3} \cdot 3$ , then pull the 3 inside the integral to make the substitution:

$\int e^{3x} dx = \frac{1}{3} \cdot 3 \int e^{3x} dx = \frac{1}{3} \int e^{3x} \cdot 3 dx = \frac{1}{3} \int e^u du = \frac{1}{3} e^u + C = \frac{1}{3} e^{3x} + C$ . Either method will get us the same answer; however, I tend to prefer method #1.

*Remember, though, this can only be used to adjust constant multipliers. It cannot be used to adjust variables!*

**Example 4.**  $\int \frac{x-1}{x^2-2x+3} dx$

Let  $u = x^2 - 2x + 3$ ,  $du = (2x - 2)dx$ . But that's not quite what we have in the numerator, But if we factor out a 2 and divide:  $(2x - 2)dx = 2(x - 1)dx = du \Rightarrow (x - 1)dx = \frac{1}{2} du$ . Substituting

now we get:  $\int \frac{x-1}{x^2-2x+3} dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x^2 - 2x + 3| + C$ .

**Example 5.**  $\int \frac{2}{e^x - 3} dx$

Sometimes, you can do algebra to get a problem that doesn't look like it can be done with substitution into a form where it can be done. Exponentials are most susceptible to this.

Here, multiply by one in the form of  $\frac{e^{-x}}{e^{-x}}$ . We use this form because exponentials need a

second copy for the derivative. If we had an  $e^{-x}$  in the denominator, we'd multiply by  $\frac{e^x}{e^x}$

instead. But in our case:  $\int \frac{2}{e^x-3} dx = \int \frac{2}{e^x-3} \cdot \frac{e^{-x}}{e^{-x}} dx = \int \frac{2e^{-x}}{1-3e^{-x}} dx$  once we distribute through the denominator. Now, we can do substitution with  $u$  equal to the denominator.

$u = 1 - 3e^{-x}, du = 3e^{-x} dx \Rightarrow \frac{1}{3} du = e^{-x} dx$  after the two negatives cancel each other out. The 2 in the numerator can just be pulled out of the integral.

$$\int \frac{2e^{-x}}{1-3e^{-x}} dx = 2 \int \frac{e^{-x} dx}{1-3e^{-x}} = \frac{2}{3} \int \frac{du}{u} = \frac{2}{3} \ln |u| + C = \frac{2}{3} \ln |1-3e^{-x}| + C$$

**Example 6.**  $\int \frac{x-4}{x^2-6x+9} dx$

Let  $u = x^2 - 6x + 9, du = (2x - 6)dx$ . The common factor on the derivative is 2 and dividing

this out we get  $\frac{1}{2} du = (x-3)dx$ , but this isn't exactly what we have. If we want to continue

doing the problem here, we have to split the problem into two parts where the numerator

adds to  $x-4$  and one piece is  $x-3$ , i.e.  $\int \frac{x-4}{x^2-6x+9} dx = \int \frac{x-3}{x^2-6x+9} dx + \int \frac{-1}{x^2-6x+9} dx$ . We

can do the first of these integrals by substitution, but the second piece will have to be done by completing the square and either doing a power rule or an inverse tangent integral or partial fractions, depending on whether the denominator is a perfect square (as it is here) or whether it has an extra constant (what would happen if our denominator was  $x^2 - 6x + 10$  instead).

To finish off the example here:

$$\begin{aligned} \int \frac{x-4}{x^2-6x+9} dx &= \int \frac{x-3}{x^2-6x+9} dx + \int \frac{-1}{x^2-6x+9} dx = \\ &= \int \frac{\frac{1}{2} du}{u} + \int \frac{-1}{(x-3)^2} dx = \frac{1}{2} \int \frac{du}{u} - \int \frac{1}{w^2} dw = \frac{1}{2} \int \frac{du}{u} - \int w^{-2} dw = \frac{1}{2} \ln |u| + w^{-1} + C = \\ &= \frac{1}{2} \ln |x^2-6x+9| + \frac{1}{x-3} + C = \ln |x-3| + \frac{1}{x-3} + C \end{aligned}$$

Where I let  $w = x - 3, dw = dx$  in the second integral, and in the last step, I just simplified using log rules.

**Practice Problems.**

1. Write each composite function  $h(x)$  as  $f(g(x))$  or  $a(b(c(x)))$  by listing  $f(x)$ ,  $g(x)$  or  $a(x)$ ,  $b(x)$ ,  $c(x)$  as appropriate. Think about this in terms of the chain rule: how many times will you have to apply it? If just once, you have the first kind of function; if twice, the second kind.

a.  $h(x) = \ln(3x)$

b.  $h(x) = 3^{(x-2)^2}$

c.  $h(x) = \ln(e^{x^2} + 1)$

d.  $h(x) = \ln^2(2x)$

e.  $h(x) = \sqrt{1 + 6x^2}$

f.  $h(x) = \ln(13x - 11)$

g.  $h(x) = (x^3 + 3x - 1)^{7/3}$

h.  $h(x) = \frac{1}{4x - 17}$

i.  $h(x) = \sqrt{1 + \sqrt{x}}$

2. For each of the  $g(x)$  and  $b(c(x))$  in a-i above, find  $g'(x)$  and  $b'(c(x))c'(x)$ .

3. Integrate.

j.  $\int \ln(3x) dx$

k.  $\int (x - 2) \cdot 3^{(x-2)^2} dx$

l.  $\int \frac{xe^{x^2} \ln(e^{x^2} + 1)}{e^{x^2} + 1} dx$

m.  $\int x\sqrt{1 + 6x^2} dx$

n.  $\int \frac{\ln(13x - 11)}{13x - 11} dx$

o.  $\int (x^2 + 1)(x^3 + 3x - 1)^{7/3} dx$

p.  $\int \frac{1}{4x - 17} dx$

q.  $\int \frac{1}{\sqrt{x}\sqrt{1 + \sqrt{x}}} dx$

4. Integrate the examples from the “Tips” section.

r.  $\int \frac{\ln x}{x} dx$

s.  $\int \frac{1}{(\log_4 p)p} dp$

t.  $\int \frac{x \ln(x^2 + 1)}{x^2 + 1} dx$

u.  $\int 2xe^{x^2+2} dx$

v.  $\int \frac{e^x \ln(e^x + 1)}{e^x + 1} dx$

w.  $\int 3x^2 e^{x^3} dx$

x.  $\int (x^3 + 2x - 6)^4 (3x^2 + 2) dx$

y.  $\int 2x\sqrt{1+x^2} dx$

z.  $\int \frac{2z}{1-z^2} dz$

aa.  $\int \frac{2x-2}{x^2-2x+1} dx$

bb.  $\int \frac{t^3}{t^4+1} dt$

cc.  $\int \frac{e^x}{e^x+3} dx$

5. Integrate.

dd.  $\int \frac{\log_5 x}{x} dx$

ee.  $\int \frac{2x-5}{1+x^2} dx$

ff.  $\int \frac{4}{5+2e^{-x}} dx$