Riemann Sums key (Practice Problems) Exact.

1. $\int_0^2 2x-5\,dx$

a.

① $\Delta x = \frac{b-a}{n} = \frac{2}{n}$ ② $\Delta x_i = 0 + \frac{2i}{n} = \frac{2i}{n}$

③ $f(x_i) = 2(\frac{2i}{n}) - 5 = \frac{4i}{n} - 5$

④ $\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} (\frac{4i}{n} - 5)\frac{2}{n} = \sum_{i=1}^{n} (\frac{8i}{n^2} - \frac{10}{n})$

⑤ $\lim_{n\to\infty} \sum (\frac{8i}{n^2} - \frac{10}{n}) = \sum \frac{8i}{n^2} - \sum \frac{10}{n} =$

$\frac{8}{n^2}\sum i - \sum \frac{10}{n} = \frac{8}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{10}{n}\cdot n =$

$\frac{8n}{2n} + \frac{8}{2n} - 10 = 4 + \frac{4}{n} - 10 = -6 + \frac{4}{n}$

⑥ $\lim_{n\to\infty} -6 + \frac{4}{n}\to 0 = -6$

by Fundamelal Theorem: $\int_0^2 2x-5\,dx = x^2 - 5x\Big|_0^2 = 4 - 10 = -6$ ✓

b. $\int_1^4 2x-5\,dx$

① $\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$ ② $\Delta x_i = 1 + \frac{3i}{n}$

③ $f(x_i) = 2(1+\frac{3i}{n}) - 5 = 2 + \frac{6i}{n} - 5 = \frac{6i}{n} - 3$

④ $\sum_{i=1}^{n} f(x_i) = (\frac{6i}{n} - 3)(\frac{2}{n}) = \sum_{i=1}^{n}(\frac{12i}{n^2} - \frac{6}{n})$

⑤ $\sum_{i=1}^{n}(\frac{12i}{n^2} - \frac{6}{n}) = \sum_{i=1}^{n} \frac{12i}{n^2} - \sum_{i=1}^{n} \frac{6}{n} = \frac{12}{n^2}\sum_{i=1}^{n} i - \sum_{i=1}^{n} \frac{6}{n}$

$= \frac{12}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{6}{n}\cdot n = \frac{6n}{n} + \frac{6}{n} - 6 =$

$6 + \frac{6}{n} - 6 = \frac{4}{n}$

⑥ $\lim_{n\to\infty} \frac{4}{n}\to 0 = 0$

by FTC: $\int_1^4 2x-5\,dx = x^2 - 5x\Big|_1^4 = (16-20) - (1-5) = -4 - (-4) = 0$

2. a. $\int_0^2 x^2+1\,dx$

① $\Delta x = \frac{2}{n}$ ② $x_i = a + \Delta x i = 0 + \frac{2i}{n} = \frac{2i}{n}$

③ $f(x_i) = \left(\frac{2i}{n}\right)^2 + 1 = \frac{4i^2}{n^2} + 1$

④ $\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(\frac{4i^2}{n^2}+1\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left(\frac{8i^2}{n^3} + \frac{2}{n}\right)$

⑤ $\frac{8i}{n^3}\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n}\frac{2}{n} = \frac{8i}{n^3}\left(\frac{n(2n+1)(n+1)}{6}\right) + \frac{2}{n}n$

$= \frac{4(2n^2+3n+1)}{3n^2} + 2 = \frac{8n^2}{3n^2} + \frac{12n}{3n^2} + \frac{4}{3n^2} + 2$

⑥ $\lim_{n\to\infty} \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} + 2 = \frac{8}{3} + 2 = \frac{14}{3}$

by FTC: $\int_0^2 x^2+1\,dx = \frac{1}{3}x^3 + x \Big|_0^2 = \frac{1}{3}(8) + 2 = \frac{14}{3}$ ✓

b. $\int_1^4 x^2+1\,dx$

① $\Delta x = \frac{4-1}{n} = \frac{3}{n}$ ② $x_i = 1 + \frac{3i}{n}$

③ $f(x_i) = \left(1+\frac{3i}{n}\right)^2 + 1 = 1 + \frac{6i}{n} + \frac{9i^2}{n^2} + 1 =$
$2 + \frac{6i}{n} + \frac{9i^2}{n^2}$

④ $\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(2 + \frac{6i}{n} + \frac{9i^2}{n^2}\right)\frac{3}{n} =$
$\sum_{i=1}^{n}\left(\frac{6}{n} + \frac{18i}{n^2} + \frac{27i^2}{n^3}\right)$

⑤ $= \sum_{i=1}^{n}\frac{6}{n} + \frac{18}{n^2}\sum_{i=1}^{n} i + \frac{27}{n^3}\sum_{i=1}^{n} i^2 =$

$\frac{6}{n}\cdot n + \frac{18}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{27}{n^3}\left[\frac{(2n+1)(n+1)(n)}{6}\right]$

$= 6 + \frac{9n}{n} + \frac{9}{n} + \frac{9}{2n^2}(2n^2+3n+1) =$

$6 + 9 + \frac{9}{n} + \frac{18n^2}{2n^2} + \frac{27n}{2n^2} + \frac{9}{2n^2} =$

$6 + 9 + \frac{9}{n} + 9 + \frac{27}{2n} + \frac{9}{2n^2} = 24 + \frac{36}{2n} + \frac{9}{2n^2}$

⑥ $\lim_{n\to\infty} 24 + \frac{18}{n} + \frac{9}{2n^2} = 24$

by FTC: $\int_1^4 x^2+1\,dx = \frac{1}{3}x^3 + x\Big|_1^4 = \frac{1}{3}(64) + 4 - \left(\frac{1}{3}(1) + 1\right) =$
$\frac{64-1}{3} + 3 = \frac{63}{3} + 3 = 21 + 3 = 24$ ✓

③ a. $\int_0^2 x^2 - x + 6\,dx$

① $\Delta x = \frac{2}{n}$ $\quad \Delta x_i = \frac{2i}{n}$

③ $f(x_i) = \left(\frac{2i}{n}\right)^2 - \frac{2i}{n} + 6 = \frac{4i^2}{n^2} - \frac{2i}{n} + 6$

④ $\sum_{i=1}^{n} \left(\frac{4i^2}{n^2} - \frac{2i}{n} + 6\right)\left(\frac{2}{n}\right) =$

$\sum_{i=1}^{n} \left(\frac{8i^2}{n^3} - \frac{4i}{n^2} + \frac{12}{n}\right)$

⑤ $\frac{8}{n^3}\sum i^2 - \frac{4}{n^2}\sum i + \frac{12}{n}\sum 1 =$

$\frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] - \frac{4}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{12}{n}\cdot n$

$= \frac{4(2n^2+3n+1)}{3n^2} - \frac{2n+2}{n} + 12 =$

$\frac{8n^2}{3n^2} + \frac{12n}{3n^2} + \frac{4}{3n^2} - \frac{2n}{n} - \frac{2}{n} + 12 =$

$\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} - 2 - \frac{2}{n} + 12 =$

$\frac{38}{3} + \frac{2}{n} + \frac{4}{3n^2}$

⑥ $\lim_{n\to\infty} \frac{38}{3} + \frac{2}{n} + \frac{4}{3n^2} = \frac{38}{3}$

by FTC $\int_0^2 x^2 - x + 6\,dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x\Big|_0^2 = \frac{8}{3} - 2 + 12 = \frac{38}{3}$ ✓

b) $\int_1^4 x^2 - x + 6\,dx$

① $\Delta x = \frac{4-1}{n} = \frac{3}{n}$ ② $\Delta x_i = a + \Delta x i = 1 + \frac{3i}{n}$

③ $f(x_i) = \left(1 + \frac{3i}{n}\right)^2 - \left(1 + \frac{3i}{n}\right) + 6 =$

$1 + \frac{6i}{n} + \frac{9i^2}{n^2} - 1 - \frac{3i}{n} + 6 = 6 + \frac{3i}{n} + \frac{9i^2}{n^2}$

④ $\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(6 + \frac{3i}{n} + \frac{9i^2}{n^2}\right)\left(\frac{3}{n}\right) =$

$\sum_{i=1}^{n}\left(\frac{18}{n} + \frac{9i}{n^2} + \frac{27i^2}{n^3}\right)$

⑤ $\sum_{i=1}^{n}\frac{18}{n} + \frac{9}{n^2}\sum_{i=1}^{n} i + \frac{27}{n^3}\sum_{i=1}^{n} i^2 =$

$\frac{18}{n}\cdot n + \frac{9}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{27}{n^3}\left[\frac{n(2n+1)(n+1)}{6}\right]$

$18 + \frac{9n}{2n} + \frac{9}{2n} + \frac{9(2n^2+3n+1)}{2n^2} =$

$18 + \frac{9}{2} + \frac{9}{2n} + \frac{18n^2}{2n^2} + \frac{27n}{2n^2} + \frac{9}{2n^2} =$

$\frac{63}{2} + \frac{36}{2n} + \frac{9}{2n^2}$ ⑥ $\lim_{n\to\infty} \frac{63}{2} + \frac{18}{n} + \frac{9}{2n^2} = \frac{63}{2}$

by FTC $\int_1^4 x^2 - x + 6\,dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x\Big|_1^4 =$

$\frac{1}{3}(64) - \frac{1}{2}(16) + 24 - \left(\frac{1}{3}(1) - \frac{1}{2}(1) + 6\right) = \frac{64}{3} + 16 - \frac{1}{3} + \frac{1}{2} - 6 =$

$21 + 10 + \frac{1}{2} = \frac{63}{2}$ ✓

1. a $\int_0^2 x^3\,dx$

① $\Delta x = \frac{2}{n}$ ② $x_i = a + \Delta x i = \frac{2i}{n}$

③ $f(x_i) = \left(\frac{2i}{n}\right)^3 = \frac{8i^3}{n^3}$ ④ $\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(\frac{8i^3}{n^3}\right)\left(\frac{2}{n}\right) = \frac{16i^3}{n^4}$

⑤ $\sum_{i=1}^{n}\frac{16i^3}{n^4} = \frac{16}{n^4}\sum i^3 = \frac{16}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] = \frac{4(n^2+2n+1)}{n^2} =$

$\frac{4n^2}{n^2} + \frac{8n}{n^2} + \frac{4}{n^2} = 4 + \frac{8}{n} + \frac{4}{n^2}$

⑥ $\lim_{n\to\infty} 4 + \frac{8}{n}^{\to 0} + \frac{4}{n^2}^{\to 0} = 4$

by FTC $\int_0^2 x^3\,dx = \frac{1}{4}x^4\Big|_0^2 = \frac{1}{4}(16) = 4$ ✓

b. $\int_1^4 x^3\,dx$

① $\Delta x = \frac{4-1}{n} = \frac{3}{n}$ ② $x_i = a + \Delta x i = 1 + \frac{3i}{n}$

③ $f(x_i) = \left(1+\frac{3i}{n}\right)^3 = 1 + \frac{27i^2}{n^2} + \frac{9i}{n} + \frac{27i^3}{n^3}$

use $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

④ $\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(1 + \frac{9i}{n} + \frac{27i^2}{n^2} + \frac{27i^3}{n^3}\right)\left(\frac{3}{n}\right) =$

$\sum_{i=1}^{n}\left(\frac{3}{n} + \frac{27i}{n^2} + \frac{81i^2}{n^3} + \frac{81i^3}{n^4}\right) =$

⑤ $\frac{3}{n}\sum_{i=1}^{n} 1 + \frac{27}{n^2}\sum_{i=1}^{n} i + \frac{81}{n^3}\sum_{i=1}^{n} i^2 + \frac{81}{n^4}\sum_{i=1}^{n} i^3 =$

$\frac{3}{n}\cdot n + \frac{27}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{81}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{81}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]$

$= 3 + \frac{27n}{2n} + \frac{27}{2n} + \frac{81}{6n^2}(2n^2+3n+1) + \frac{81}{4n^2}(n^2+2n+1)$

$= 3 + \frac{27}{2} + \frac{27}{2n} + \frac{54n^2}{2n^2} + \frac{81n}{2n^2} + \frac{27}{2n^2} + \frac{81n^2}{4n^2} + \frac{162n}{4n^2} + \frac{81}{4n^2}$

$= 3 + \frac{27}{2} + \frac{27}{2n} + 27 + \frac{81}{2n} + \frac{27}{2n^2} + \frac{81}{4} + \frac{81}{2n} + \frac{81}{4n^2} =$

$= \frac{255}{4} + \frac{189}{2n} + \frac{135}{4n^2}$

⑥ $\lim_{n\to\infty} \frac{255}{4} + \frac{189}{2n}^{\to 0} + \frac{135}{4n^2}^{\to 0} = \frac{255}{4}$

by FTC $\int_1^4 x^3\,dx = \frac{1}{4}x^4\Big|_1^4 = \frac{1}{4}(256) - \frac{1}{4}(1) = \frac{256}{4} - \frac{1}{4} =$

$\frac{255}{4}$ ✓

⑤ cannot be done exactly this way
when we get to integration by parts, you'll be able to get an exact answer by FTC.

1.a. for right hand rule Step #2 = $\Delta x_i = a + x_i$ Numerical

1.a. from exact calculation, the end of Step #5 gives

$\approx -6 + \frac{4}{n}$. plug n=5 & n=6 in here.

$n=5 \quad \approx -6 + \frac{4}{5} = \frac{-26}{5} = -5.2$

$n=6 \quad \approx -6 + \frac{4}{6} = -\frac{16}{3} = -5.3\overline{3}$ These are getting closer to the true answer.

b. similarly for [1,4] step 5 yields $\approx \frac{4}{n}$

$n=5 \quad \approx \frac{4}{5} = .8$

$n=6 \quad \approx \frac{4}{6} = \frac{2}{3} \approx .66$

2a. $\approx \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} + 2 = \frac{14}{3} + \frac{4}{n} + \frac{4}{3n^2}$

$n=5 \approx \frac{14}{3} + \frac{4}{5} + \frac{4}{75} \approx 5.47$

$n=6 \approx \frac{14}{3} + \frac{4}{6} + \frac{4}{108} \approx 5.37$

2b. $\approx 24 + \frac{18}{n} + \frac{9}{2n^2}$

$n=5 \quad \approx 24 + \frac{18}{5} + \frac{9}{50} \approx 27.78$

$n=6 \quad \approx 24 + \frac{18}{6} + \frac{9}{72} \approx 27.125$

3a. $\approx \frac{38}{3} + \frac{2}{n} + \frac{4}{3n^2}$

$n=5 \quad \approx \frac{38}{3} + \frac{2}{5} + \frac{4}{3 \cdot 25} \approx 13.12$

$n=6 \quad \approx \frac{38}{3} + \frac{2}{6} + \frac{4}{108} \approx 13.037$

3b. $\approx \frac{63}{2} + \frac{18}{n} + \frac{9}{2n^2} =$

$n=5 \quad \approx \frac{63}{2} + \frac{18}{5} + \frac{9}{50} = 35.28$

$n=6 \quad \approx \frac{63}{2} + \frac{18}{6} + \frac{9}{72} = 34.625$

4a. $\approx 4 + \frac{8}{n} + \frac{4}{n^2}$

$n=5 \approx 4 + \frac{8}{5} + \frac{4}{25} = 5.76$

$n=6 \approx 4 + \frac{8}{6} + \frac{4}{36} \approx 5.4\overline{4}$

b. $\approx \frac{255}{4} + \frac{189}{2n} + \frac{135}{4n^2}$ $n=5 \approx \frac{255}{4} + \frac{189}{5} + \frac{135}{100} = 102.9$; $n=6 \approx \frac{255}{4} + \frac{189}{6} + \frac{135}{144} \approx 95.6875$

5. $\int_0^2 \ln(x+1)\,dx$

① $\Delta x = \frac{2}{n}$

② $\Delta x_i = a + \Delta x i$ (right-hand rule)
$= 0 + \frac{2i}{n} = \frac{2i}{n}$

③ $f(x_i) = \ln\left(\frac{2i}{n}+1\right)$

④ $\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \ln\left(\frac{2i}{n}+1\right)\left(\frac{2}{n}\right) = \frac{2}{n}\sum_{i=1}^{n} \ln\left(\frac{2i}{n}+1\right)$

for $n=5$

$\frac{2}{5}\sum_{i=1}^{5} \ln\left(\frac{2i}{5}+1\right) = \frac{2}{5}\left[\ln\left(\frac{2}{5}+1\right) + \ln\left(\frac{4}{5}+1\right) + \ln\left(\frac{6}{5}+1\right) + \ln\left(\frac{8}{5}+1\right) + \ln(3)\right]$

$\approx \frac{2}{5}\left[\ln(7/5) + \ln(9/5) + \ln(11/5) + \ln(13/5) + \ln(3)\right]$

≈ 1.50674

for $n=6$

$\frac{2}{6}\sum_{i=1}^{6} \left(\ln(2i/6+1)\right) =$

$\frac{2}{6}\left[\ln\left(\frac{2}{6}+1\right) + \ln\left(\frac{4}{6}+1\right) + \ln\left(\frac{6}{6}+1\right) + \ln\left(\frac{8}{6}+1\right) + \ln\left(\frac{10}{6}+1\right) + \ln(3)\right] =$

$= \frac{1}{3}\left[\ln(4/3) + \ln(5/3) + \ln(2) + \ln(7/3) + \ln(8/3) + \ln(3)\right]$

≈ 1.4728

b. $\int_1^4 \ln(x+1)\,dx$

① $\Delta x = \frac{4-1}{n} = \frac{3}{n}$ ② $x_i = a + \Delta x i = 1 + \frac{3i}{n}$

③ $f(x_i) = \ln\left(1+\frac{3i}{n}+1\right) = \ln\left(2+\frac{3i}{n}\right)$

④ $\sum_{i=1}^{n} f(x_i)\Delta x = \sum \frac{3}{n}\left(\ln\left(2+\frac{3i}{n}\right)\right) = \frac{3}{n}\sum \ln\left(2+\frac{3i}{n}\right)$

for $n=5$

$\frac{3}{5}\sum_{i=1}^{5} \ln\left(2+\frac{3i}{5}\right) =$

$\frac{3}{5}\left[\ln\left(2+\frac{3}{5}\right) + \ln\left(2+\frac{6}{5}\right) + \ln\left(2+\frac{9}{5}\right) + \ln\left(2+\frac{12}{5}\right) + \ln(5)\right]$

$= \frac{3}{5}\left[\ln\left(\frac{13}{5}\right) + \ln\left(\frac{16}{5}\right) + \ln\left(\frac{19}{5}\right) + \ln\left(\frac{22}{5}\right) + \ln(5)\right] \approx 3.92682$

for $n=6$

$\frac{3}{6}\sum_{i=1}^{6} \ln\left(2+\frac{3i}{6}\right) = \frac{1}{2}\sum_{i=1}^{6} \ln\left(2+\frac{i}{2}\right) \approx$

$\frac{1}{2}\left[\ln\left(2+\frac{1}{2}\right) + \ln(3) + \ln\left(2+\frac{3}{2}\right) + \ln(4) + \ln\left(2+\frac{5}{2}\right) + \ln(5)\right]$

$= \frac{1}{2}\left[\ln(5/2) + \ln(3) + \ln(7/2) + \ln(4) + \ln(9/2) + \ln(5)\right]$

≈ 3.88374

Left-endpoint rule $\quad x_i = a + \Delta x(i-1) \quad$ step #2

1a. $\int_0^2 2x-5\,dx$

① $\Delta x = \frac{2}{n}$ ② $x_{i-1} = 0 + \frac{2}{n}(i-1) = \frac{2i}{n} - \frac{2}{n}$

③ $f(x_{i-1}) = 2\left(\frac{2i}{n} - \frac{2}{n}\right) - 5 = \frac{4i}{n} - \frac{4}{n} - 5$

④ $\sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n}\left(\frac{4i}{n} - \frac{4}{n} - 5\right)\left(\frac{2}{n}\right) = \frac{8i}{n^2} - \frac{8}{n^2} - \frac{10}{n}$

⑤ $\frac{8}{n^2}\sum i - \sum\left(\frac{8}{n^2} + \frac{10}{n}\right) = \frac{8}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{8}{n^2}\cdot n - \frac{10}{n}\cdot n$

$= \frac{4n}{n} + \frac{4}{n} - \frac{8}{n} - 10 = 4 - \frac{4}{n} - 10 = -6 - \frac{4}{n}$

$n = 5 \quad \approx -6 - \frac{4}{5} = -6.8$

$n = 6 \quad \approx -6 - \frac{4}{6} \approx -6.67$

b. $\int_1^4 2x-5\,dx$

① $\Delta x = \frac{4-1}{n} = \frac{3}{n}$ ② $x_{i-1} = a + \Delta x(i-1) = 1 + \frac{3i}{n} - \frac{3}{n}$

③ $f(x_{i-1}) = 2\left(1 + \frac{3i}{n} - \frac{3}{n}\right) - 5 = 2 + \frac{6i}{n} - \frac{6}{n} - 5 = -3 + \frac{6i}{n} - \frac{6}{n}$

④ $\sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum\left(-3 + \frac{6i}{n} - \frac{6}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left(-\frac{9}{n} + \frac{18i}{n^2} - \frac{18}{n^2}\right)$

⑤ $\sum_{i=1}^{n} -\frac{9}{n} + \frac{18}{n^2}\sum_{i=1}^{n} i - \sum_{i=1}^{n}\frac{18}{n^2} = -\frac{9}{n}\cdot n + \frac{18}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{18}{n^2}\cdot n$

$= -9 + \frac{9n}{n} + \frac{9}{n} - \frac{18}{n} = 0 - \frac{9}{n}$

$n = 5 \quad \approx \frac{-9}{5} = -1.8$

$n = 6 \quad \approx \frac{-9}{6} = -1.5$

2a. $\int_0^2 x^2+1\,dx$

① $\Delta x = \frac{2}{n}$ ② $x_{i-1} = 0 + \frac{2(i-1)}{n} = \frac{2i}{n} - \frac{2}{n}$

③ $f(x_{i-1}) = \left(\frac{2i}{n} - \frac{2}{n}\right)^2 + 1 = \frac{4i^2}{n^2} - \frac{8i}{n^2} + \frac{4}{n^2} + 1$

④ $\sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n}\left(\frac{4i^2}{n^2} - \frac{8i}{n^2} + \frac{4}{n^2} + 1\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n} \frac{8i^2}{n^3} - \frac{16i}{n^3} + \frac{8}{n^2} + \frac{2}{n}$

⑤ $\frac{8}{n^3}\sum_{i=1}^{n} i^2 - \frac{16}{n^3}\sum_{i=1}^{n} i + \sum_{i=1}^{n}\left(\frac{8}{n^2} + \frac{2}{n}\right) = \frac{8}{n^3}\left[\frac{n(2n+1)(n+1)}{6}\right] - \frac{16}{n^3}\left[\frac{n(n+1)}{2}\right] + \frac{8}{n^2}\cdot n + \frac{2}{n}\cdot n$ $\quad (2n^2+3n+1)$

$\frac{8n^2}{3n^2} + \frac{12n}{3n^2} + \frac{4}{3n^2} - \frac{8n}{n^2} - \frac{8}{n^2} + \frac{8}{n} + 2 = \frac{14}{3} + \frac{4}{n} + \frac{4}{3n^2} - \frac{8}{n^2}$

2a continued

$$\approx \frac{14}{3} + \frac{4}{n} + \frac{4}{3n^2} - \frac{24}{3n^2} = \frac{14}{3} + \frac{4}{n} + \frac{20}{3n^2}$$

$$n=5 \approx \frac{14}{3} + \frac{4}{5} - \frac{20}{75} = 5.2$$

$$n=6 \approx \frac{14}{3} + \frac{4}{6} - \frac{20}{108} \approx 5.14815$$

2b. $\int_1^4 x^2+1\,dx$ ① $\Delta x = \frac{3}{n}$ ② $x_{i-1} = a + \Delta x(i-1) = 1 + \frac{3}{n}(i-1) = 1 + \frac{3i}{n} - \frac{3}{n}$

③ $f(x_{i-1}) = \left(1+\frac{3i}{n}-\frac{3}{n}\right)^2 + 1 = \left(1+\frac{3i}{n}-\frac{3}{n}\right)\left(1+\frac{3i}{n}-\frac{3}{n}\right)+1 =$

$$1 + \frac{3i}{n} - \frac{3}{n} + \frac{3i}{n} + \frac{9i^2}{n^2} - \frac{9i}{n^2} - \frac{3}{n} - \frac{9i}{n^2} + \frac{9}{n^2} + 1 =$$

$$2 + \frac{6i}{n} - \frac{6}{n} + \frac{9i^2}{n^2} - \frac{18i}{n^2} + \frac{9}{n^2} =$$

$$\left(2 - \frac{6}{n} + \frac{9}{n^2}\right) + \left(\frac{6}{n} - \frac{18}{n^2}\right)i + \frac{9i^2}{n^2}$$

④ $\sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n}\left[\left(2-\frac{6}{n}+\frac{9}{n^2}\right) + \left(\frac{6}{n}-\frac{18}{n^2}\right)i + \frac{9i^2}{n^2}\right]\left(\frac{3}{n}\right) =$

$$\sum_{i=1}^{n}\left(\frac{6}{n} - \frac{18}{n^2} + \frac{27}{n^3}\right) + \left(\frac{18}{n^2} - \frac{54}{n^3}\right)\sum_{i=1}^{n} i + \frac{27}{n^3}\sum i^2$$

⑤ $= \left(\frac{6}{n} - \frac{18}{n^2} + \frac{27}{n^3}\right)n + \left(\frac{18}{n^2} - \frac{54}{n^3}\right)\left[\frac{n(n+1)}{2}\right] + \frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$

$$\left(6 - \frac{18}{n} + \frac{27}{n^2}\right) + \left(\frac{9n}{n} + \frac{9}{n} - \frac{27n}{n^2} - \frac{27}{n^2}\right) + \left(\frac{18n^2}{2n^2} + \frac{27n}{2n^2} + \frac{9}{2n^2}\right)$$

$$= 6 - \frac{18}{n} + \frac{27}{n^2} + 9 + \frac{9}{n} - \frac{27}{n} - \frac{27}{n^2} + 9 + \frac{27}{2n} + \frac{9}{2n^2}$$

$$= 24 - 45/2n + \frac{9}{2n^2}$$

$$n=5 \approx 24 - \frac{45}{10} + \frac{9}{50} = 19.68$$

$$n=6 \approx 24 - \frac{45}{12} + \frac{9}{72} \approx 20.375$$

3a $\int_0^2 x^2 - x + 6\,dx$ ① $\Delta x = \frac{2}{n}$ ② $x_{i-1} = 0 + \frac{2}{n}(i-1) = \frac{2i}{n} - \frac{2}{n}$

③ $f(x_{i-1}) = \left(\frac{2i}{n} - \frac{2}{n}\right)^2 - \left(\frac{2i}{n} - \frac{2}{n}\right) + 6 =$

$$\frac{4i^2}{n^2} - \frac{8i}{n^2} + \frac{4}{n^2} - \frac{2i}{n} + \frac{2}{n} + 6$$

④ $\sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n}\left(\frac{4i^2}{n^2} - \frac{8i}{n^2} + \frac{4}{n^2} - \frac{2i}{n} + \frac{2}{n} + 6\right)\left(\frac{2}{n}\right)$

$$\sum_{i=1}^{n}\left(\frac{8i^2}{n^3} - \frac{16i}{n^3} + \frac{8}{n^2} - \frac{4i}{n^2} + \frac{4}{n^2} + \frac{12}{n}\right) =$$

$$\left(\frac{8}{n^3}\right)\sum i^2 + \left(\frac{-16}{n^3} - \frac{4}{n^2}\right)\sum i + \left(\frac{8}{n^2} + \frac{4}{n^2} + \frac{12}{n}\right)\sum 1 =$$

$$\frac{\overset{4}{8}}{n^{\not 3}_2}\left[\frac{\not n(n+1)\overset{2n^2+3n+1}{(2n+1)}}{\not 6_3}\right] + \left(-\frac{\overset{8}{\not 16}}{n^{\not 3 2}} - \frac{\overset{2}{\not 4}}{n^{\not 2}}\right)\left[\frac{\not n(n+1)}{\not 2}\right] + \left(\frac{8}{n^{\not 2}} + \frac{4}{n^{\not 2}} + \frac{12}{\not n}\right)\not n$$

$$\frac{8n^{\not 2}}{3n^{\not 2}} + \frac{\overset{4}{\not 12 \not n}}{\not 3 n^{\not 2}} + \frac{4}{3n^2} - \frac{8\not n}{n^{\not 2}} - \frac{8}{n^2} - \frac{2\not n}{\not n} - \frac{2}{n} + \frac{8}{n} + \frac{4}{n} + 12$$

$$\left(\frac{8}{3} - 2 + 12\right) + \left(\frac{4}{n} - \frac{2}{n} + \frac{12}{n}\right) + \left(\frac{4}{3n^2} - \frac{8}{n^2}\right) =$$

$$\frac{38}{3} + \frac{14}{n} + \frac{-20}{3n^2}$$

$n=5 \approx \frac{38}{3} + \frac{14}{5} - \frac{20}{75} \approx 15.2$

$n=6 \approx \frac{38}{3} + \frac{14}{6} - \frac{20}{108} \approx 14.8148$

3b. $\int_1^4 x^2 - x + 6\,dx$ ① $\Delta x = \frac{3}{n}$ ② $x_{i-1} = a + \Delta x(i-1) = 1 + \frac{3}{n}(i-1) =$
$1 + \frac{3i}{n} - \frac{3}{n}$

③ $f(x_{i-1}) = \left(1 + \frac{3i}{n} - \frac{3}{n}\right)^2 - \left(1 + \frac{3i}{n} - \frac{3}{n}\right) + 6$

see previous problem 2b for work on $\left(1 + \frac{3i}{n} - \frac{3}{n}\right)^2$

$$= \left(\not 1 + \frac{6i}{n} - \frac{6}{n} + \frac{6i}{n^2} - \frac{18i}{n^2} + \frac{9i^2}{n^2} + \frac{9}{n^2} - \not 1 + \frac{-3i}{n} + \frac{3}{n} + 6\right)$$

$$\left[\left(6 - \frac{3}{n} + \frac{9}{n^2}\right) + \left(\frac{3}{n} + \frac{-12}{n^2}\right)i + \frac{9i^2}{n^2}\right]$$

④ $\sum_{i=1}^{n} f(x_{i-1})\Delta x =$

$$\sum_{i=1}^{n}\left[\left(6 - \frac{3}{n} + \frac{9}{n^2}\right) + \left(\frac{3}{n} - \frac{12}{n^2}\right)i + \frac{9i^2}{n^2}\right]\left(\frac{3}{n}\right) =$$

$$\sum_{i=1}^{n}\left[\left(\frac{18}{n} - \frac{9}{n^2} + \frac{27}{n^3}\right) + \left(\frac{9}{n^2} - \frac{36}{n^3}\right)i + \frac{27i^2}{n^3}\right] =$$

$$\left(\frac{18}{\not n} - \frac{9}{n^{\not 2}} + \frac{27}{n^{\not 3}_2}\right)\not n + \left(\frac{9}{\not n^{\not 2}} - \frac{36}{n^{\not 3}_2}\right)\left[\frac{\not n(n+1)}{2}\right] + \frac{\overset{9}{\not 27}}{n^{\not 3}_2}\left[\frac{\not n(n+1)\overset{2n^2+3n+1}{(2n+1)}}{\not 6_2}\right]$$

$$18 - \frac{9}{n} + \frac{27}{n^2} + \frac{9\not n}{2\not n} + \frac{9}{2n} - \frac{18\not n}{n^2} - \frac{18}{n^2} + \frac{\overset{9}{\not 18}n^{\not 2}}{\not 2 n^{\not 2}} + \frac{27\not n}{2n^{\not 2}} + \frac{9}{2n^2}$$

$$\frac{63}{2} + \frac{9}{n} + \frac{27}{2n^2}$$

$n=5 \approx \frac{63}{2} + \frac{9}{5} + \frac{27}{50} = 33.84$

$n=6 \approx \frac{63}{2} + \frac{9}{6} + \frac{27}{72} \approx 33.375$

4a. $\int_0^2 x^3\,dx$ ① $\Delta x = \frac{2}{n}$ ② $a + \Delta x(i-1) = \frac{2i}{n} - \frac{2}{n} = \frac{2}{n}(i-1)$

③ $f(x_{i-1}) = \left(\frac{2i}{n} - \frac{2}{n}\right)^3 = \left(\frac{2}{n}\right)^3 (i-1) = \frac{8}{n^3}(i^3 - 3i^2 + 3i - 1)$

$= \frac{8i^3}{n^3} - \frac{24i^2}{n^3} + \frac{24i}{n^3} - \frac{8}{n^3}$

④ $\sum_{i=1}^{n} f(x_{i-1})\Delta x = \left(\sum_{i=1}^{n}\left(\frac{8i^3}{n^3} - \frac{24i^2}{n^3} + \frac{24i}{n^3} - \frac{8}{n^3}\right)\right)\frac{2}{n} =$

$\sum_{i=1}^{n} \frac{16i^3}{n^4} - \frac{48i^2}{n^4} + \frac{48i}{n^4} - \frac{16}{n^4} =$

⑤ $\frac{16}{n^4}\sum_{i=1}^{n} i^3 - \frac{48}{n^4}\sum_{i=1}^{n} i^2 + \frac{48}{n^4}\sum_{i=1}^{n} i - \sum_{i=1}^{n}\frac{16}{n^4}$

$\frac{\cancel{16}^{4}}{\cancel{n^4}_{2}}\left[\frac{\cancel{n^2}(n+1)^2}{\cancel{4}}\right] - \frac{\cancel{48}^{8}}{\cancel{n^4}_{3}}\left[\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}}\right] + \frac{\cancel{48}^{24}}{\cancel{n^4}_{3}}\left(\frac{\cancel{n}(n+1)}{\cancel{2}}\right) - \frac{16}{\cancel{n^4}_{3}}\cdot\cancel{n}$

(n^2+2n+1; $2n^2+3n+1$)

$= \frac{4\cancel{n^2}}{\cancel{n^2}} + \frac{8n}{n^2} + \frac{4}{n^2} - \frac{16n^2}{n^3} - \frac{24n}{n^3} - \frac{8}{n^3} + \frac{24n}{n^3} + \frac{24}{n^3} - \frac{16}{n^3}$

$= 4 + \frac{8}{n} + \frac{16}{n} + \frac{4}{n^2} - \cancel{\frac{24}{n^2}} + \cancel{\frac{24}{n^2}} - \cancel{\frac{8}{n^3}} + \cancel{\frac{24}{n^3}} - \cancel{\frac{16}{n^3}}$

$= 4 + \frac{24}{n} + \frac{4}{n^2}$

$n=5 \approx 4 + \frac{24}{5} + \frac{4}{25} = 8.96$

$n=6 \approx 4 + \frac{24}{6} + \frac{4}{36} \approx 8.11$

4b. $\int_1^4 x^3\,dx$ ① $\Delta x = \frac{3}{n}$ ② $x_{i-1} = 1 + \frac{3}{n}(i-1) = 1 + \frac{3i}{n} - \frac{3}{n}$

③ $f(x_{i-1}) = \left(1 + \frac{3i}{n} - \frac{3}{n}\right)^3 =$

$\frac{27i^3}{n^3} + \frac{27i^2}{n^2} - \frac{81i^2}{n^3} + \frac{9i}{n} - \frac{54i}{n^2} + \frac{81i}{n^3} - \frac{9}{n} + \frac{27}{n^2} - \frac{27}{n^3} + 1$

④ $\sum_{i=1}^{n} f(x_{i-1})\Delta x =$

$\sum_{i=1}^{n}\left(\frac{27i^3}{n^3} + \frac{27i^2}{n^2} - \frac{81i^2}{n^3} + \frac{9i}{n} - \frac{54i}{n^2} + \frac{81i}{n^3} - \frac{9}{n} + \frac{27}{n^2} + \frac{27}{n^3} + 1\right)\left(\frac{3}{n}\right) =$

$\sum_{i=1}^{n}\left(\frac{81i^3}{n^4} + \frac{81i^2}{n^3} - \frac{243i^2}{n^4} + \frac{27i}{n^2} - \frac{162i}{n^3} + \frac{243i}{n^4} - \frac{27}{n^2} + \frac{81}{n^3} + \frac{81}{n^4} + \frac{3}{n}\right)$

$$\frac{81}{n^4}\sum_{i=1}^{n} i^3 + \left(\frac{81}{n^3}-\frac{243}{n^4}\right)\sum_{i=1}^{n} i^2 + \left(\frac{27}{n^2}-\frac{162}{n^3}+\frac{243}{n^4}\right)\sum_{i=1}^{n} i + \sum_{i=1}^{n}\left(-\frac{27}{n^2}+\frac{81}{n^3}+\frac{81}{n^4}+\frac{3}{n}\right)$$

$$= \frac{81}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] + \left(\frac{81}{n^3}-\frac{243}{n^4}\right)\left(\frac{n(n+1)(2n+1)}{6}\right) + \left(\frac{27}{n^2}-\frac{162}{n^3}+\frac{243}{n^4}\right)\left(\frac{n(n+1)}{2}\right) + n\left(\frac{27}{n^2}+\frac{81}{n^3}+\frac{81}{n^4}+\frac{3}{n}\right)$$

$$= \frac{81n^2}{4n^2} + \frac{162n}{2\cdot 4n^2} + \frac{81}{4n^2} + \frac{54n^2}{2n^2} + \frac{81n}{2n^2} + \frac{27}{n^2} - \frac{162n^2}{n^3} - \frac{243n}{2n^3} - \frac{81}{2n^3} + \frac{27n}{2n} + \frac{27}{2n} - \frac{162n}{2n^2} - \frac{162}{2n^2} + \frac{243n}{2n^3} + \frac{243}{2n^3} + \frac{27}{n} + \frac{81}{n^2} + \frac{81}{n^3} + 3$$

$$= \frac{81}{4} + \frac{81}{2n} + \frac{81}{4n^2} + 27 + \frac{81}{2n} + \frac{27}{n^2} - \frac{162}{n} - \frac{243}{2n^2} - \frac{81}{2n^3} + \frac{27}{2} + \frac{27}{2n} - \frac{81}{n} - \frac{81}{n^2} + \frac{243}{2n^2} + \frac{243}{2n^3} + \frac{27}{n} + \frac{81}{n^2} + \frac{81}{n^3} + 3$$

$$= \left(\frac{81}{4}+27+\frac{27}{2}+3\right) + \frac{1}{n}\left(\frac{81}{2}+\frac{81}{2}-162+\frac{27}{2}-81+27\right) + \frac{1}{n^2}\left(\frac{81}{4}+27-\frac{243}{2}-81+\frac{243}{2}+81\right) + \frac{1}{n^3}\left(-\frac{81}{2}+\frac{243}{2}+81\right)$$

$$= \frac{255}{4} + -\frac{243}{2n} + \frac{189}{4n^2} + \frac{162}{n^3}$$

$n=5 \quad \approx \frac{255}{4} - \frac{243}{10} + \frac{189}{100} + \frac{162}{125} \approx 42.636$

$n=6 \quad \approx \frac{255}{4} - \frac{243}{12} + \frac{189}{144} + \frac{162}{216} \approx 45.5625$

5a. $\int_0^2 \ln(x+1)$ ① $\Delta x = \frac{2}{n}$ ② $x_{i-1} = 0 + \frac{2(i-1)}{n} = \frac{2i}{n} - \frac{2}{n}$

③ $f(x_{i-1}) = \ln\left(1+\frac{2i}{n}-\frac{2}{n}\right)$

④ $\sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n}\left[\ln\left(1+\frac{2i}{n}-\frac{2}{n}\right)\left(\frac{2}{n}\right)\right]$

$n=5 \quad \frac{2}{5}\left[\ln\left(1+\frac{2}{5}-\frac{2}{5}\right) + \ln\left(1+\frac{2}{5}\right) + \ln\left(1+\frac{4}{5}\right) + \ln\left(1+\frac{6}{5}\right) + \ln\left(1+\frac{8}{5}\right)\right]$

$= \frac{2}{5}\left[\ln(7/5) + \ln(9/5) + \ln(11/5) + \ln(13/5)\right]$

≈ 1.06729

$n=6 \quad \frac{1}{3}\left[\ln(1) + \ln\left(1+\frac{1}{3}\right) + \ln\left(1+\frac{2}{3}\right) + \ln(2) + \ln\left(1+\frac{4}{3}\right) + \ln\left(1+\frac{5}{3}\right)\right]$

$= \frac{1}{3}\left[\ln(4/3) + \ln(5/3) + \ln(2) + \ln(7/3) + \ln(8/3)\right] \approx 1.10659$

5b. $\int_1^4 \ln(x+1)\,dx$ ① $\Delta x = \frac{3}{n}$ ② $x_{i-1} = 1 + \frac{3}{n}(i-1) = 1 + \frac{3i}{n} - \frac{3}{n}$

③ $f(x_{i-1}) = \ln\left(1 + \frac{3i}{n} - \frac{3}{n}\right)$

④ $\sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} \ln\left(1+\frac{3i}{n}-\frac{3}{n}\right)\left(\frac{3}{n}\right)$

$n=5$

$\frac{3}{5}\left[\ln(2) + \ln\left(2+\frac{3}{5}\right) + \ln\left(2+\frac{6}{5}\right) + \ln\left(2+\frac{9}{5}\right) + \ln\left(2+\frac{12}{5}\right)\right]$

$= \frac{3}{5}\left[\ln(2) + \ln\left(\frac{13}{5}\right) + \ln\left(\frac{16}{5}\right) + \ln\left(\frac{19}{5}\right) + \ln\left(\frac{22}{5}\right)\right] =$

≈ 3.37705

$n=6$

$\frac{1}{2}\left[\ln(2) + \ln\left(\frac{5}{2}\right) + \ln(3) + \ln\left(\frac{7}{2}\right) + \ln(4) + \ln\left(\frac{9}{2}\right)\right] =$

≈ 3.42559

Midpoint Rule

1a. $\int_0^2 2x-5\,dx$ ① $\Delta x = \frac{2}{n}$ $\frac{1}{2}\Delta x = \frac{1}{n}$

$x_i = \frac{1}{n}, \frac{3}{n}, \frac{5}{n}, \frac{7}{n}, \ldots\; 2-\frac{1}{n}$

$\sum_{i=1}^{n} f(x_i)\Delta x =$

$n=5$ $\frac{2}{5}\left[\left(2\left(\frac{1}{5}\right)-5\right) + \left(2\left(\frac{3}{5}\right)-5\right) + \left(2\left(\frac{5}{5}\right)-5\right) + \left(2\left(\frac{7}{5}\right)-5\right) + \left(2\left(\frac{9}{5}\right)-5\right)\right]$

$= \frac{2}{5}\left[2\left(\frac{1}{5}+\frac{3}{5}+1+\frac{7}{5}+\frac{9}{5}\right) - 5\times 5\right] =$

$= \frac{2}{5}\left[2\left(\frac{25}{5}\right) - 25\right] = \frac{2}{5}[10-25] = \frac{2}{5}[-15] = -6$ ✓

$n=6$ $\frac{1}{3}\left[2\left(\frac{1}{6}\right)-5 + 2\left(\frac{3}{6}\right)-5 + 2\left(\frac{5}{6}\right)-5 + 2\left(\frac{7}{6}\right)-5 + 2\left(\frac{9}{6}\right)-5 + 2\left(\frac{11}{6}\right)-5\right] =$

$\frac{1}{3}\left[2\left(\frac{1}{6}+\frac{3}{6}+\frac{5}{6}+\frac{7}{6}+\frac{9}{6}+\frac{11}{6}\right) - 5\times 6\right] = \frac{1}{3}\left[12 - 30\right] = \frac{1}{3}[-18] = -6$ ✓

The midpoint rule is just the trapezoidal rule in disguise and is accurate everywhere the trapezoidal rule is.

$\int_1^4 2x-5\,dx$ ① $\Delta x = \frac{3}{n}$ $\frac{1}{2}\Delta x = \frac{3}{2n}$

$x_i = 1+\frac{3}{2n},\ 1+\frac{9}{2n},\ 1+\frac{15}{2n}, \dots\ 4-\frac{3}{2n}$

$\sum_{i=1}^{n} f(x_i)\Delta x =$

$n=5$

$\frac{3}{5}\left[2(1+\frac{3}{10})-5+2(1+\frac{9}{10})-5+2(1+\frac{15}{10})-5+2(1+\frac{21}{10})-5+2(1+\frac{27}{5})-5\right]$

$=\frac{3}{5}\left[2\left(\frac{13}{10}+\frac{19}{10}+\frac{25}{10}+\frac{31}{10}+\frac{37}{10}\right)-5\times5\right]=$

$=\frac{3}{5}\left[2\left(\frac{125}{10}\right)-25\right]=\frac{3}{5}[25-25]=\frac{3}{5}[0]=0$

$n=6$

$\frac{1}{2}\left[2(\frac{5}{4})-5+2(\frac{7}{4})-5+2(\frac{9}{4})-5+2(\frac{11}{4})-5+2(\frac{13}{4})-5+2(\frac{15}{4})-5\right]$

$=\frac{1}{2}\left[2\left(\frac{5}{4}+\frac{7}{4}+\frac{9}{4}+\frac{11}{4}+\frac{13}{4}+\frac{15}{4}\right)-5\times6\right]=\frac{1}{2}[30-30]=0$

2a. $\int_0^2 x^2+1\,dx$ ① $\Delta x = \frac{2}{n}$ $\frac{1}{2}\Delta x = \frac{1}{n}$

$x_i = \frac{1}{n},\ \frac{3}{n},\ \frac{5}{n}, \dots\ 2-\frac{1}{n}$

$\sum_{i=1}^{n} f(x_i)\Delta x =$

$n=5$

$\frac{2}{5}\left[(\frac{1}{5})^2+1+(\frac{3}{5})^2+1+(\frac{5}{5})^2+1+(\frac{7}{5})^2+1+(\frac{9}{5})^2+1\right]=$

$\frac{2}{5}\left[\frac{1}{25}+\frac{9}{25}+\frac{49}{25}+\frac{81}{25}+6\times1\right]=\frac{2}{5}\left[\frac{140}{25}+6\right]=\frac{116}{25}=4.64$

$n=6$

$\frac{1}{3}\left[(\frac{1}{6})^2+1+(\frac{1}{2})^2+1+(\frac{5}{6})^2+1+(\frac{7}{6})^2+1+(\frac{3}{2})^2+1+(\frac{11}{6})^2+1\right]=$

$\frac{1}{3}\left[\frac{1}{36}+\frac{1}{4}+\frac{25}{36}+\frac{49}{36}+\frac{9}{4}+\frac{121}{36}+6\right]=\frac{1}{3}\left[\frac{251}{18}\right]=\frac{251}{54}\approx4.64815$

2b. $\int_1^4 x^2+1\,dx$ ① $\Delta x = \frac{3}{n}$ $\frac{1}{2}\Delta x = \frac{3}{2n}$

$x_i = 1+\frac{3}{2n},\ 1+\frac{9}{2n},\ 1+\frac{15}{2n}, \dots,\ 4-\frac{3}{2n}$

$$\sum_{i=1}^{n} f(x_i)\Delta x =$$

$n=5$

$$\frac{3}{5}\left[\left(1+\frac{3}{10}\right)^2+1+\left(1+\frac{9}{10}\right)^2+1+\left(1+\frac{15}{10}\right)^2+1+\left(1+\frac{21}{10}\right)^2+1+\left(1+\frac{27}{10}\right)^2+1\right]$$

$$=\frac{3}{5}\left[1.3^2+1.9^2+2.5^2+3.1^2+3.7^2+6\right]\approx 24.51$$

$n=6$

$$\frac{1}{2}\left[\left(1+\frac{1}{4}\right)^2+1+\left(1+\frac{3}{4}\right)^2+1+\left(1+\frac{5}{4}\right)^2+1+\left(1+\frac{7}{4}\right)^2+1+\left(1+\frac{9}{4}\right)^2+1+\left(1+\frac{11}{4}\right)^2+1\right]$$

$$=\frac{1}{2}\left[1.25^2+1.75^2+2.25^2+2.75^2+3.25^2+3.75^2+6\right]$$

$$\approx 23.9375$$

3a. $\int_0^2 x^2-x+6\,dx$ ① $\Delta x=\frac{2}{n}$ $\frac{1}{2}\Delta x=\frac{1}{n}$

$$x_i=\frac{1}{n},\frac{3}{n},\frac{5}{n},\ldots 2-\frac{1}{n}$$

$$\sum_{i=1}^{n} f(x_i)\Delta x =$$

$n=5$ $$\frac{2}{5}\left[\left(\frac{1}{5}\right)^2-\left(\frac{1}{5}\right)+6+\left(\frac{3}{5}\right)^2-\left(\frac{3}{5}\right)+6+\left(\frac{5}{5}\right)^2-\left(\frac{5}{5}\right)+6+\left(\frac{7}{5}\right)^2-\left(\frac{7}{5}\right)+6+\left(\frac{9}{5}\right)^2-\left(\frac{9}{5}\right)+6\right]=$$

$$\frac{2}{5}\left[\left(\frac{1}{25}+\frac{9}{25}+\frac{25}{25}+\frac{49}{25}+\frac{81}{25}\right)-\left(\frac{1}{5}+\frac{3}{5}+\frac{5}{5}+\frac{7}{5}+\frac{9}{5}\right)+6\times5\right]=$$

$$\frac{2}{5}\left[\frac{165}{25}-\left(\frac{25}{5}\right)+30\right]=\frac{316}{25}=12.64$$

$n=6$ $$\frac{1}{3}\left[\left(\frac{1}{6}\right)^2-\left(\frac{1}{6}\right)+6+\left(\frac{3}{6}\right)^2-\left(\frac{3}{6}\right)+6+\left(\frac{5}{6}\right)^2-\left(\frac{5}{6}\right)+6+\left(\frac{7}{6}\right)^2-\left(\frac{7}{6}\right)+6+\left(\frac{9}{6}\right)^2-\left(\frac{9}{6}\right)+6+\left(\frac{11}{6}\right)^2-\left(\frac{11}{6}\right)+6\right]=$$

$$\frac{1}{3}\left[\left(\frac{1}{36}+\frac{9}{36}+\frac{25}{36}+\frac{49}{36}+\frac{81}{36}+\frac{121}{36}\right)-\left(\frac{1}{6}+\frac{3}{6}+\frac{5}{6}+\frac{7}{6}+\frac{9}{6}+\frac{11}{6}\right)+6\times6\right]=$$

$$\frac{1}{3}\left[\frac{286}{36}-\frac{36}{6}+36\right]=\frac{683}{54}\approx 12.6481$$

3b. $\int_1^4 x^2-x+6\,dx$ ① $\Delta x=\frac{3}{n}$ $\frac{1}{2}\Delta x=\frac{3}{2n}$

$$x_i=1+\frac{3}{2n},1+\frac{9}{2n},1+\frac{15}{2n},\ldots 4-\frac{3}{2n}$$

$$\sum_{i=1}^{n} f(x_i)\Delta x =$$

$n=5$ $\frac{3}{5}[(1.3)^2 - 1.3 + 6 + 1.9^2 - 1.9 + 6 + 2.5^2 - 2.5 + 6 + 3.1^2 - 3.1$
$+ 6 + 3.7^2 - 3.7 + 6] =$

$\frac{3}{5}[1.3^2 + 1.9^2 + 2.5^2 + 3.1^2 + 3.7^2 - (1.3 + 1.9 + 2.5 + 3.1 + 3.7) + 6 \times 5]$

$= \frac{3}{5}[34.85 - 12.5 + 30] = 31.41$

$n=6$

$\frac{1}{2}[(1+\frac{1}{12})^2 - (1+\frac{1}{12}) + 6 + (1+\frac{9}{12})^2 - (1+\frac{9}{12}) + 6 + (1+\frac{15}{12})^2 - (1+\frac{15}{12}) + 6 +$
$(1+\frac{21}{12})^2 - (1+\frac{21}{12}) + 6 + (1+\frac{27}{12})^2 - (1+\frac{27}{12}) + 6 + (1+\frac{33}{12})^2 - (1+\frac{33}{12}) + 6]$

$= \frac{1}{2}[1 \times 6 - 1 \times 6 + \frac{2}{12} - \frac{1}{12} + \frac{18}{12} - \frac{9}{12} + \frac{30}{12} - \frac{15}{12} + \frac{42}{12} - \frac{21}{12} + \frac{54}{12} - \frac{27}{12} + \frac{66}{12} - \frac{33}{12} +$
$\frac{1}{144} + \frac{81}{144} + \frac{225}{144} + \frac{441}{144} + \frac{729}{144} + \frac{1089}{144} + 6 \times 6]$

$= \frac{1}{2}[\frac{1}{12} + \frac{9}{12} + \frac{15}{12} + \frac{21}{12} + \frac{27}{12} + \frac{33}{12} + 36 + \frac{2566}{144}] = \frac{1}{2}[\frac{106}{12} + 36 + \frac{2566}{144}] =$

$= \frac{2249}{72} \approx 31.2361$

4a. $\int_0^2 x^3\,dx$ ① $\Delta x = \frac{2}{n}$ $\frac{1}{2}\Delta x = \frac{1}{n}$

$x_i = \frac{1}{n}, \frac{3}{n}, \frac{5}{n}, \ldots 2 - \frac{1}{n}$

$n=5$ $\sum_{i=}^{n} f(x_i)\Delta x =$

$\frac{2}{5}[(\frac{1}{5})^3 + (\frac{3}{5})^3 + (\frac{5}{5})^3 + (\frac{7}{5})^3 + (\frac{9}{5})^3] =$

$\frac{2}{5}[\frac{1}{125} + \frac{27}{125} + \frac{125}{125} + \frac{343}{125} + \frac{729}{125}] = \frac{2}{5}[\frac{1255}{125}] = \frac{502}{125} = 4.016$

$n=6$
$\frac{1}{3}[(\frac{1}{6})^3 + (\frac{3}{6})^3 + (\frac{5}{6})^3 + (\frac{7}{6})^3 + (\frac{9}{6})^3 + (\frac{11}{6})^3] =$

$\frac{1}{3}[\frac{1}{216} + \frac{27}{216} + \frac{125}{216} + \frac{343}{216} + \frac{729}{216} + \frac{1331}{216}] = \frac{431}{108} \approx 3.99074$

4b $\int_1^4 x^3\,dx$ ① $\Delta x = \frac{3}{n}$ $\frac{1}{2}\Delta x = \frac{3}{2n}$

$x_i = 1 + \frac{3}{2n}, 1 + \frac{9}{2n}, 1 + \frac{15}{2n}, \ldots 4 - \frac{3}{2n}$

$\sum_{i=1}^{n} f(x_i)\Delta x =$

$n=5$

$$\frac{3}{5}\left[(1.3)^3+1.9^3+2.5^3+3.1^3+3.7^3\right]= \quad 63.075$$

$n=6$

$$\frac{1}{2}\left[\left(1+\frac{3}{12}\right)^3+\left(1+\frac{9}{12}\right)^3+\left(1+\frac{15}{12}\right)^3+\left(1+\frac{21}{12}\right)^3+\left(1+\frac{27}{12}\right)^3+\left(1+\frac{33}{12}\right)^3\right]$$

$$\quad \tfrac{1}{4} \qquad \tfrac{3}{4} \qquad \tfrac{5}{4} \qquad \tfrac{7}{4} \qquad \tfrac{9}{4} \qquad \tfrac{11}{4}$$

$$\frac{1}{2}\left[1.25^3+1.75^3+2.25^3+2.75^3+3.25^3+3.75^3\right]=$$

$$=63.2813$$

5a. $\int_0^2 \ln(x+1)\,dx \qquad \Delta x=\frac{2}{n} \qquad \frac{1}{2}\Delta x=\frac{1}{n}$

$$x_i=\frac{1}{n},\ \frac{3}{n},\ \frac{5}{n},\ \dots\ 2-\frac{1}{n}$$

$$\sum_{i=1}^{n} f(x_i)\Delta x=$$

$n=5$ $\frac{2}{5}\left[\ln\left(1+\frac{1}{5}\right)+\ln\left(1+\frac{3}{5}\right)+\ln\left(1+\frac{5}{5}\right)+\ln\left(1+\frac{7}{5}\right)+\ln\left(1+\frac{9}{5}\right)\right]=$

$\frac{2}{5}\left[\ln(1.2)+\ln(1.6)+\ln(2)+\ln(2.4)+\ln(2.8)\right]= \quad 1.30022$

$n=6$ $\frac{1}{3}\left[\ln\left(1+\frac{1}{6}\right)+\ln\left(1+\frac{3}{6}\right)+\ln\left(1+\frac{5}{6}\right)+\ln\left(1+\frac{7}{6}\right)+\ln\left(1+\frac{9}{6}\right)+\ln\left(1+\frac{11}{6}\right)\right]$

$=\frac{1}{3}\left[\ln(7/6)+\ln(1.5)+\ln(11/6)+\ln(13/6)+\ln(2.5)+\ln(17/6)\right]$

$=\ 1.2989$

5b. $\int_1^4 \ln(x+1)\,dx \qquad \Delta x=\frac{3}{n} \qquad \frac{1}{2}\Delta x=\frac{3}{2n}$

$$x_i=1+\frac{3}{2n},\ 1+\frac{9}{2n},\ 1+\frac{15}{2n},\ \dots,\ 4-\frac{3}{2n}$$

$n=5$ $\quad \sum_{i=1}^{n} f(x_i)\Delta x$

$$\frac{3}{5}\left[\ln(2.3)+\ln(2.9)+\ln(3.5)+\ln(4.1)+\ln(4.7)\right]= \quad 3.66536$$

$n=6$ $\frac{1}{2}\left[\ln\left(1+\frac{3}{12}\right)+\ln\left(1+\frac{9}{12}\right)+\ln\left(1+\frac{15}{12}\right)+\ln\left(1+\frac{21}{12}\right)+\ln\left(1+\frac{27}{12}\right)+\ln\left(1+\frac{33}{12}\right)\right]=$

$\frac{1}{2}\left[\ln(2.25)+\ln(2.75)+\ln(3.25)+\ln(3.75)+\ln(4.25)+\ln(4.75)\right]=$

$=3.66400$