Trig Substitution

Trig substitution is generally used to evaluate integrals that have squared terms under radicals that do not conform to simple inverse trig rules. (The two in question are $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x)+C$, and $\int \frac{1}{x\sqrt{x^2-1}}dx = \text{arcsec}(x)+C$. However, these, too, can be integrated this way.) Trig substitution depends on making a substitution involving trigonometric functions which, through Pythagorean identities, we are able to simplify the square root and integrate a simpler function. Afterwards, of course, we have to return to the original variable.

There are three basic trig substitutions:

1. When the radical is of the form $\sqrt{a^2-u^2}$, we substitute $u = a\sin\theta$.
2. When the radical is of the form $\sqrt{a^2+u^2}$, we substitute $u = a\tan\theta$.
3. When the radical if of the form $\sqrt{u^2-a^2}$, we substitute $u = a\sec\theta$.

You'll notice that the forms of the radicals in each substitution are the same form as those in the derivate of each inverse trig function. These forms are chosen because of the forms of the Pythagorean identities which will help us reduce.

Examples.

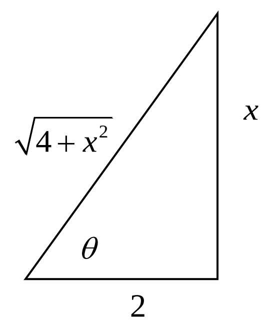
a. Let's try this with $\int \frac{1}{\sqrt{1-x^2}}dx$. We know what the answer is, but we can confirm that we get the same result with this technique. We are going to make the substitution $x = 1\sin\theta$. We will also need $dx = \cos\theta d\theta$. If we substitute into the integral we get: $\int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta d\theta$. By applying the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$, we get $\int \frac{1}{\sqrt{\cos^2\theta}}\cos\theta d\theta = \int \frac{\cos\theta d\theta}{\cos\theta}$ $= \int d\theta = \theta + C$. Now we need to switch back to x. We do that by going back to our substitution. If we solve for $\theta \to \theta = \arcsin(x)$, thus our integral is $\arcsin(x)+C$.

b. Let's try a trickier one. $\int \frac{x^3}{\sqrt{4+x^2}}dx$. This one we can't do by traditional substitution. Here we choose $x = 2\tan\theta$ as our substitution. And $dx = 2\sec^2\theta d\theta$. Putting this into our integral, we get $\int \frac{x^3}{\sqrt{4+x^2}}dx = \int \frac{8\tan^3\theta \cdot 2\sec^2\theta d\theta}{\sqrt{4+4\tan^2\theta}}$ Factor out the 4 under the square root and apply the Pythagorean identity $\tan^2\theta + 1 = \sec^2\theta$:

$= \int \frac{16\tan^3\theta\sec^2\theta d\theta}{\sqrt{4\left(1+\tan^2\theta\right)}} = \int \frac{16\tan^3\theta\sec^2\theta d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{16\tan^3\theta\sec^2\theta d\theta}{2\sec\theta} = \int 8\tan^3\theta\sec\theta d\theta$. To integrate this, we will isolate $\sec\theta\tan\theta d\theta$ as du, and we'll convert the remaining $\tan^2\theta$ using the Pythagorean identity we used before. Thus:

$\int 8\tan^3\theta\sec\theta d\theta = 8\int\left(\sec^2\theta - 1\right)\left(\sec\theta\tan\theta d\theta\right)$. Let $u = \sec\theta$ to integrate: $8\int u^2 - 1du$

$8\left[\frac{u^3}{3} - u\right] + C$, but now we have to convert back to $\theta \to 8\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C$. But we still aren't done yet. We have to go all the way back to x. To do this, we need to draw a triangle based on our original substitution. Solve for $\tan\theta = \frac{x}{2}$, and use the definition of tangent as "opposite over adjacent" to place the pieces on the triangle. The Pythagorean theorem gives the third side. Turns out it will always be the original radical from our problem.



Use the triangle to find the replacement for $\sec\theta$.

For this triangle $\sec\theta = \frac{\sqrt{4+x^2}}{2}$.

Substitute this back into the solution to our integral.

$$\frac{8}{3}\left(\frac{\sqrt{4+x^2}}{2}\right)^3 - 8\left(\frac{\sqrt{4+x^2}}{2}\right) + C$$

This is our final solution.

c. Let's try one more. $\int x\left(x^2-7\right)^{5/2}dx$. Remember that rational exponents are just radicals in disguise. The substitution we will make here is $x = \sqrt{7}\sec\theta$, and $dx = \sqrt{7}\sec\theta\tan\theta d\theta$. Replacing in our equation we get:

$$\int x\left(x^2-7\right)^{5/2}dx = \int \sqrt{7}\sec\theta\left(7\sec^2 - 7\right)^{5/2}\sqrt{7}\sec\theta\tan\theta d\theta =$$

$$7\int \sec^2\theta\tan\theta\left[7\left(\sec^2\theta - 1\right)\right]^{5/2}d\theta = 7\int \sec^2\theta\tan\theta\left[7\left(\tan^2\theta\right)\right]^{5/2}d\theta$$

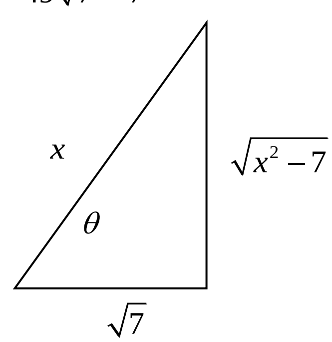
$$7\int \sec^2\theta\tan\theta \cdot 49\sqrt{7}\tan^5\theta d\theta = 343\sqrt{7}\int \tan^6\theta\sec^2\theta d\theta$$

$49\sqrt{7} = 7^{5/2}$

From here, we let $u = \tan\theta, du = \sec^2\theta d\theta$. This gives us:

$$343\sqrt{7}\int u^6 du = 49\sqrt{7}u^7 + C = 49\sqrt{7}\tan^7\theta + C$$

Our triangle will get us back to the original x.



$$49\sqrt{7}\left(\frac{\sqrt{x^2-7}}{\sqrt{7}}\right)^7 + C = \frac{1}{7}\left(x^2-7\right)^{7/2} + C$$

Problems.

i. Use trig substitution to verify that $\int \frac{1}{x\sqrt{x^2-1}}dx = \text{arcsec}\, x + C$.

ii. $\int x^2\sqrt{1-x^2}\,dx$

iii. $\int x^2\left(1+x^2\right)^{3/2}dx$

iv. $\int \frac{x}{\sqrt{2x-x^2}}dx$ [Hint: complete the square.]

v. $\int \frac{1}{\sqrt{x^2+8x+19}}dx$ [Hint: complete the square.]

vi. $\int \frac{x^2}{\sqrt{x^2-6x-5}}dx$ [Hint: complete the square.]

vii. $\int \frac{x^5}{\sqrt{8-x^2}}dx$

viii. $\int \frac{3e^{4x}}{\left(9-e^{2x}\right)^{3/2}}dx$

ix. $\int \frac{x^6}{\sqrt{1+x^4}}dx$