Homework # 6 Math 268 Spring 2012

1. i)

$$\begin{bmatrix} .35-\lambda & .25 \\ -.35 & 1.25-\lambda \end{bmatrix} \quad p = .35$$

$$(.35-\lambda)(1.25-\lambda) + .35 * .25 = 0$$

$$\lambda_1 = .4608835 \quad \lambda_2 = 1.1391165$$

$$\begin{bmatrix} .35-.46 & .25 \\ -.35 & 1.25-.46 \end{bmatrix} \quad -.11x_1 = -.25x_2$$

$$x_1 = 2.\overline{27}x_2 \approx {}^{25}\!/_{11}\, x_2 \qquad V_1 = \begin{bmatrix} 25 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} .35-1.139 & .25 \\ -.35 & 1.25-1.139 \end{bmatrix} \quad -.789x_1 = -.25x_2$$

$$x_1 = .3168x_2 \qquad V_2 = \begin{bmatrix} 1 \\ 3.156 \end{bmatrix}$$

will approach $V_2$ over the long run

ratio cats: .24
chipmunks .76

ii) $p = .5$

$$\begin{bmatrix} .35-\lambda & .25 \\ -.5 & 1.25-\lambda \end{bmatrix}$$

$$(.35-\lambda)(1.25-\lambda) + .5 * .25 = 0$$

$$\lambda_1 = .52 \quad \lambda_2 = 1.07$$

$$\begin{bmatrix} .35-.52 & .25 \\ -.5 & 1.25-.52 \end{bmatrix} \quad -.17x_1 = -.25x$$

$$\lambda_1 = 1.47x_2 \qquad V_1 = \begin{bmatrix} 1 \\ .68 \end{bmatrix}$$

$$\begin{bmatrix} .35-1.07 & .25 \\ -.5 & 1.25-1.07 \end{bmatrix} \quad -.72x_1 = -.25x_2$$

$$x_1 = .3472 \qquad V_2 = \begin{bmatrix} 1 \\ 2.88 \end{bmatrix}$$

will approach $V_2$ over the long run

ratio: cats: .257
chipmunks: .74

iii) $p = .7$

$$\begin{bmatrix} .35-\lambda & .25 \\ -.7 & 1.25 \end{bmatrix}$$

$$(.35-\lambda)(1.25-\lambda) + .7 * .25 = 0$$

$$\lambda_1 = .63 \qquad \lambda_2 = .965$$

$$\begin{bmatrix} -.63+.35 & .25 \\ -.7 & 1.25-.63 \end{bmatrix} \qquad -.28x_1 = -.25x_2 \qquad v_1 = \begin{bmatrix} 1 \\ 1.12 \end{bmatrix}$$

$$x_1 = .89x_2$$

$$\begin{bmatrix} .35-.965 & .25 \\ -.7 & 1.25-.965 \end{bmatrix} \qquad -.615x_1 = -.25x_2 \qquad v_2 = \begin{bmatrix} 1 \\ 2.46 \end{bmatrix}$$

$$x_1 = .4x_2$$

both $\lambda$ are less than zero so population will collapse to zero along both eigenvectors.

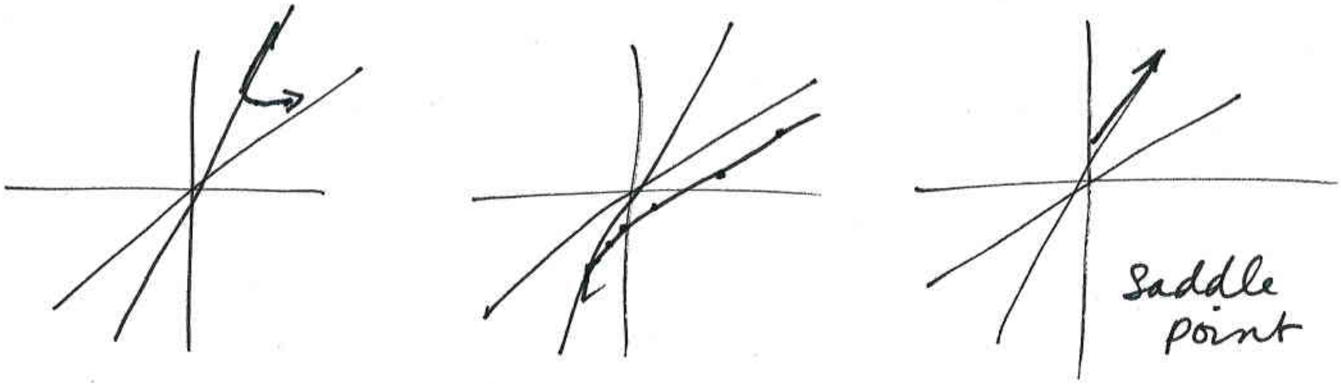
b.

i) $\begin{bmatrix} .35 & .25 \\ -.35 & 1.25 \end{bmatrix} \qquad x_0 = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$

$$x_k = \begin{bmatrix} 10 \\ 15 \end{bmatrix} \begin{bmatrix} 7.25 \\ 15.25 \end{bmatrix} \begin{bmatrix} 6.35 \\ 16.5 \end{bmatrix} \begin{bmatrix} 6.3 \\ 18.4 \end{bmatrix} \begin{bmatrix} 6.8 \\ 20.8 \end{bmatrix} \begin{bmatrix} 7.6 \\ 23.6 \end{bmatrix} \begin{bmatrix} 8.56 \\ 26.88 \end{bmatrix}$$

$$y_k = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 1.3 \\ .2 \end{bmatrix} \begin{bmatrix} .505 \\ -.285 \end{bmatrix} \begin{bmatrix} .1255 \\ -.433 \end{bmatrix} \begin{bmatrix} -.064 \\ -.585 \end{bmatrix} \begin{bmatrix} -.1688 \\ -.708 \end{bmatrix} \begin{bmatrix} -.236 \\ -.827 \end{bmatrix}$$

$$z_k = \begin{bmatrix} 2 \\ 10 \end{bmatrix} \begin{bmatrix} 3.2 \\ 11.8 \end{bmatrix} \begin{bmatrix} 4.07 \\ 13.63 \end{bmatrix} \begin{bmatrix} 4.83 \\ 15.6 \end{bmatrix} \begin{bmatrix} 5.6 \\ 17.8 \end{bmatrix} \begin{bmatrix} 6.4 \\ 20.3 \end{bmatrix} \begin{bmatrix} 7.3 \\ 23.2 \end{bmatrix}$$

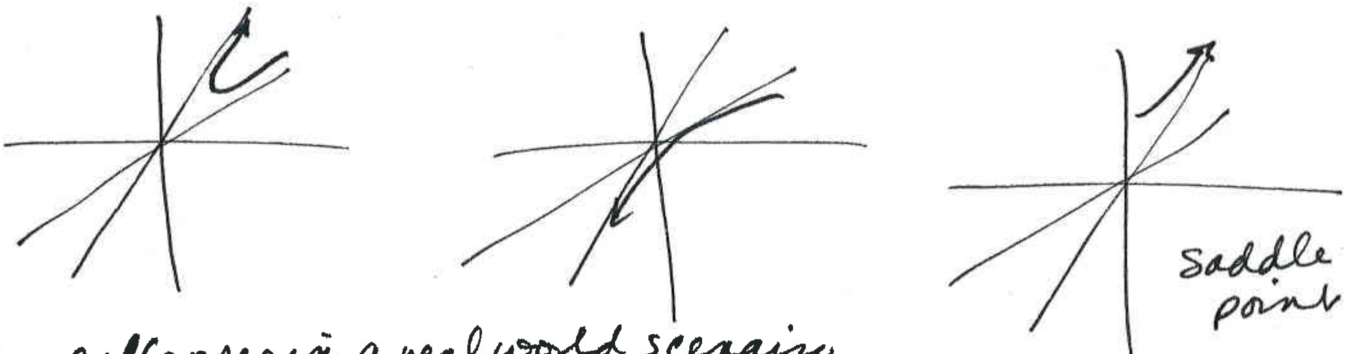


in a realworld scenario $y_k$ would collapse since it approaches the eigenvector in the neg. direction.

ii) $x_k = \begin{bmatrix} 10 \\ 15 \end{bmatrix} \begin{bmatrix} 7.25 \\ 13.75 \end{bmatrix} \begin{bmatrix} 5.97 \\ 13.56 \end{bmatrix} \begin{bmatrix} 5.48 \\ 13.96 \end{bmatrix} \begin{bmatrix} 5.4 \\ 14.7 \end{bmatrix} \begin{bmatrix} 5.57 \\ 15.7 \end{bmatrix} \begin{bmatrix} 5.87 \\ 16.8 \end{bmatrix}$

$y_k = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 1.3 \\ -.25 \end{bmatrix} \begin{bmatrix} .39 \\ -.96 \end{bmatrix} \begin{bmatrix} -.10 \\ -1.399 \end{bmatrix} \begin{bmatrix} -.38 \\ -1.69 \end{bmatrix} \begin{bmatrix} -.56 \\ -1.9 \end{bmatrix} \begin{bmatrix} -.68 \\ -2.13 \end{bmatrix}$

$z_k = \begin{bmatrix} 2 \\ 10 \end{bmatrix} \begin{bmatrix} 3.2 \\ 11.5 \end{bmatrix} \begin{bmatrix} 4 \\ 12.78 \end{bmatrix} \begin{bmatrix} 4.59 \\ 13.97 \end{bmatrix} \begin{bmatrix} 5.10 \\ 15.168 \end{bmatrix} \begin{bmatrix} 5.58 \\ 16.4 \end{bmatrix} \begin{bmatrix} 6.05 \\ 17.7 \end{bmatrix}$

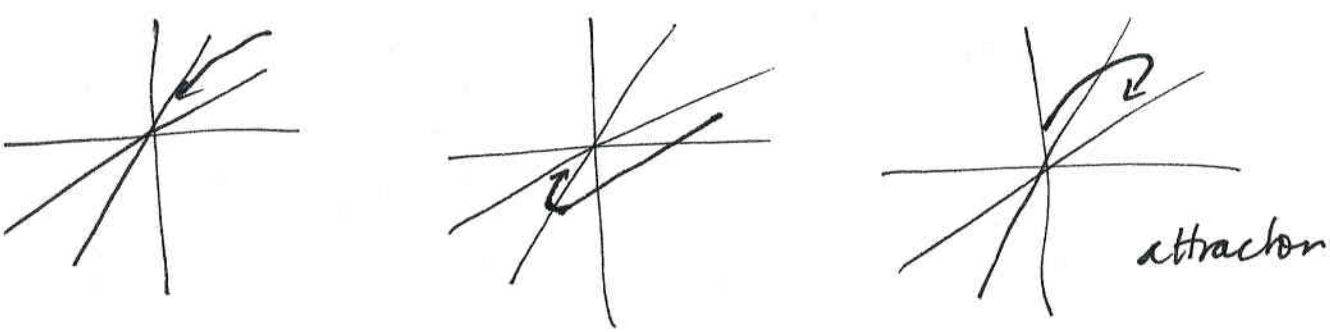



$y_k$ also collapses in a realworld scenario

iii) $x_k = \begin{bmatrix} 10 \\ 15 \end{bmatrix} \begin{bmatrix} 7.25 \\ 11.75 \end{bmatrix} \begin{bmatrix} 5.47 \\ 9.6 \end{bmatrix} \begin{bmatrix} 4.3 \\ 8.18 \end{bmatrix} \begin{bmatrix} 3.56 \\ 7.2 \end{bmatrix} \begin{bmatrix} 3.05 \\ 6.5 \end{bmatrix} \begin{bmatrix} 2.69 \\ 6.01 \end{bmatrix} \begin{bmatrix} 2.4 \\ 5.6 \end{bmatrix}$

$y_k = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 1.3 \\ -.85 \end{bmatrix} \begin{bmatrix} .24 \\ -1.9 \end{bmatrix} \begin{bmatrix} -.4 \\ -2.6 \end{bmatrix} \begin{bmatrix} -.8 \\ -3.0 \end{bmatrix} \begin{bmatrix} -1.03 \\ -3.2 \end{bmatrix} \begin{bmatrix} -1.16 \\ -3.27 \end{bmatrix} \begin{bmatrix} -1.2 \\ -3.28 \end{bmatrix}$

$z_k = \begin{bmatrix} 2 \\ 10 \end{bmatrix} \begin{bmatrix} 3.2 \\ 11.1 \end{bmatrix} \begin{bmatrix} 3.89 \\ 11.6 \end{bmatrix} \begin{bmatrix} 4.3 \\ 11.8 \end{bmatrix} \begin{bmatrix} 4.4 \\ 11.8 \end{bmatrix} \begin{bmatrix} 4.5 \\ 11.6 \end{bmatrix} \begin{bmatrix} 4.47 \\ 11.36 \end{bmatrix} \begin{bmatrix} 4.4 \\ 11.07 \end{bmatrix}$




all populations collapse but some will grow temporarily before collapse

2. i) $\vec{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

a) $-1 * 4 + 2 * 6 = -4 + 12 = 8$

b) $\|\vec{u}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$ $\|\vec{v}\| = \sqrt{4^2 + 6^2} = \sqrt{16 + 36} = \sqrt{52}$

c) $\frac{\vec{u}}{\|u\|} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ $\frac{\vec{v}}{\|v\|} = \begin{bmatrix} 4/\sqrt{52} \\ 6/\sqrt{52} \end{bmatrix}$

d) $\|\vec{u}\|^2 + \|\vec{v}\|^2 = 5 + 52 = 57$

e) $\|\vec{u} + \vec{v}\|^2 = \left[\sqrt{(-1+4)^2 + (2+6)^2}\right]^2 = \left[\sqrt{3^2 + 8^2}\right]^2 = \left[\sqrt{9 + 64}\right]^2 = \left[\sqrt{73}\right]^2 = 73$

f) $\frac{u \cdot v}{v \cdot v} \vec{v} = \frac{8}{52} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \frac{2}{13} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8/13 \\ 12/13 \end{bmatrix}$

g) $\|u - v\| = \sqrt{(-1-4)^2 + (2-6)^2} = \sqrt{5^2 + 4^2} = \sqrt{25 + 16} = \sqrt{41}$

ii) $\vec{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

a) $12 \cdot 2 + 3 \times -3 + -5 \times 3 = 24 - 9 - 15 = 0$

b) $\|\vec{u}\| = \sqrt{144 + 9 + 25} = \sqrt{178}$ $\|\vec{v}\| = \sqrt{4 + 9 + 9} = \sqrt{22}$

c) $\frac{\vec{u}}{\|u\|} = \begin{bmatrix} 12/\sqrt{178} \\ 3/\sqrt{178} \\ -5/\sqrt{178} \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2/\sqrt{22} \\ -3/\sqrt{22} \\ 3/\sqrt{22} \end{bmatrix}$

d) $178 + 22 = 200$

e) $\vec{u} + \vec{v} = \begin{bmatrix} 14 \\ 0 \\ -2 \end{bmatrix}$ $\|u + v\|^2 = \left[\sqrt{196 + 4}\right]^2 = \left[\sqrt{200}\right]^2 = 200$

f) $\frac{0}{22} = 0$

5

g) $\vec{u}-\vec{v} = \begin{bmatrix} 10 \\ 6 \\ -8 \end{bmatrix}$ $\|\vec{u}-\vec{v}\| = \sqrt{100+36+64} = \sqrt{200}$

iii) $\vec{u} = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$

a) $3*-4 + 2\times 1 + -5*-2 + 0\times 6 = -12+2+10+0=0$

b) $\|\vec{u}\| = \sqrt{9+4+25+0} = \sqrt{38}$ $\|\vec{v}\| = \sqrt{16+1+4+36} = \sqrt{57}$

c) $\dfrac{\vec{u}}{\|u\|} = \begin{bmatrix} 3/\sqrt{38} \\ 2/\sqrt{38} \\ -5/\sqrt{38} \\ 0 \end{bmatrix}$ $\dfrac{\vec{v}}{\|v\|} = \begin{bmatrix} -4/\sqrt{57} \\ 1/\sqrt{57} \\ -2/\sqrt{57} \\ 6/\sqrt{57} \end{bmatrix}$

d) $38+57=95$

e) $\vec{u}+\vec{v} = \begin{bmatrix} -1 \\ 3 \\ -7 \\ 6 \end{bmatrix}$ $\|\vec{u}+\vec{v}\|^2 = 1+9+49+36=95$

f) $\dfrac{0}{57} = 0$

g) $\vec{u}-\vec{v} = \begin{bmatrix} 7 \\ 1 \\ -3 \\ -6 \end{bmatrix}$ $\|\vec{u}-\vec{v}\| = \sqrt{49+1+9+36} = \sqrt{95}$

3. a) see answers in 2g

b) parts ii) and iii) are

c) i) $\dfrac{8}{52} = \dfrac{2}{13}$ $\cos^{-1}(2/13) = 1.146$ radians $\approx 81°$

ii) \$ iii) are 90°

d) see part 2c

3 e. see answers in 2f

g. ii)

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 12 & 3 & -5 \\ 2 & -3 & 3 \end{vmatrix} = (9-15)\hat{i} - (36+10)\hat{j} + (-36-6)\hat{k}$$

$$-6\hat{i} - 46\hat{j} - 42\hat{k}$$

$u_3 = \begin{bmatrix} -6 \\ -46 \\ -42 \end{bmatrix}$ or any scalar multiple of it like $\begin{bmatrix} 3 \\ 23 \\ 21 \end{bmatrix}$

iii) $3a + 2b - 5c + 0d = 0$
$-4a + b - 2c + 6d = 0$

$\begin{bmatrix} 3 & 2 & -5 & 0 \\ -4 & 1 & -2 & 6 \end{bmatrix}$ rref $\Rightarrow$

$\begin{bmatrix} 1 & 0 & -1/11 & -12/11 \\ 0 & 1 & -26/11 & 18/11 \end{bmatrix}$

$x_1 = 1/11\, x_3 + 12/11\, x_4$
$x_2 = 26/11\, x_3 - 18/11\, x_4$

$\begin{bmatrix} 13 \\ 8 \\ 11 \\ 11 \end{bmatrix}$

$3a + 2b - 5c + 0d = 0$
$-4a + b - 2c + 6d = 0$
$13a + 8b + 11c + 11d = 0$

$\begin{bmatrix} 3 & 2 & -5 & 0 \\ -4 & 1 & -2 & 6 \\ 13 & 8 & 11 & 11 \end{bmatrix}$ rref $\Rightarrow$

$\begin{bmatrix} 1 & 0 & 0 & -19/18 \\ 0 & 1 & 0 & 23/9 \\ 0 & 0 & 1 & 7/18 \end{bmatrix}$

$x_1 = 19/18\, x_4$
$x_2 = -23/9\, x_4$
$x_3 = -7/18\, x_4$

$\begin{bmatrix} 19 \\ -46 \\ -7 \\ 18 \end{bmatrix}$

$u_3 = \begin{bmatrix} 13 \\ 8 \\ 11 \\ 11 \end{bmatrix}$ $\quad u_4 = \begin{bmatrix} 19 \\ -46 \\ -7 \\ 18 \end{bmatrix}$

f. use the unit vectors in 2c for ii) & iii).

4. $\vec{u}_1 \cdot \vec{u}_2 = 3\times 2 + -3\times 2 + 0\times -1 = 0$
$\vec{u}_2 \cdot \vec{u}_3 = 2\times 1 + 2\times 1 + (-1)\times 4 = 0$
$\vec{u}_1 \cdot \vec{u}_3 = 3\times 1 - 3\times 1 + 0\times 4 = 0$

all vectors are orthogonal

$\begin{bmatrix} 3 & 2 & 1 \\ -3 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Spans space & are lin. independent $\Rightarrow$ bases

4 cont'd the orthonormal basis are unit vectors

$$\|u_1\| = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2} \qquad \hat{u}_1 = \begin{bmatrix} 3/3\sqrt{2} \\ -3/3\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\|u_2\| = \sqrt{4+4+1} = \sqrt{9} = 3 \qquad \hat{u}_2 = \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

$$\|u_3\| = \sqrt{1+1+16} = \sqrt{18} = 3\sqrt{2} \qquad \hat{u}_3 = \begin{bmatrix} 1/3\sqrt{2} \\ 1/3\sqrt{2} \\ 4/3\sqrt{2} \end{bmatrix}$$

$$\left.\begin{matrix}\hat{u}_1\\ \hat{u}_2\\ \hat{u}_3\end{matrix}\right\} = B$$

$\vec{u}_j \cdot \vec{u}_j = 1$ in an orthonormal basis since all vectors are of length 1

$\vec{X} \cdot \vec{u}_i = c_i$

$$\vec{X}\cdot\vec{u}_1 = 5/\sqrt{2} + 3/\sqrt{2} + 0 = 8/\sqrt{2}$$

$$\vec{X}\cdot\vec{u}_2 = 10/3 - 6/3 - 1/3 = \frac{3}{3} = 1$$

$$\vec{X}\cdot\vec{u}_3 = \frac{5}{3\sqrt{2}} - \frac{3}{3\sqrt{2}} + \frac{4}{3\sqrt{2}} = \frac{6}{3\sqrt{2}} = \frac{2}{\sqrt{2}}$$

$$[X]_B = \begin{bmatrix} 8/\sqrt{2} \\ 1 \\ 2/\sqrt{2} \end{bmatrix}$$

5. $\text{proj}_{\vec{u}}\ \frac{u\cdot y}{u\cdot u} u = \frac{14+6}{49+1}\vec{u} = \frac{20}{50}\vec{u} = \frac{2}{5}\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} = \vec{y}_{\parallel}$

$$\vec{y}_{\perp} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$$

6. a) true

b) true

c) false normalized just means length 1 the direction hasn't changed

d) true

e) false – true if the matrix is $n \times n$

f) true

g) true

6h) true

i) true

j) false. the formula given is for $\vec{y}_\perp$. the best approximation is $\vec{y}_\parallel$ or yest $\text{proj}_W \vec{y}$

k) true

l) true

m) true

n) true

7. a) $\vec{u}_1 \cdot \vec{u}_2 = -2+2-1+1=0$

$\vec{u}_2 \cdot \vec{u}_3 = -2+1+2-1=0$

$\vec{u}_3 \cdot \vec{u}_4 = -1+1-2+2=0$

$\vec{u}_1 \cdot \vec{u}_3 = 1+2-2-1=0$

$\vec{u}_1 \cdot \vec{u}_4 = -1+2+-2=0$

$\vec{u}_2 \cdot \vec{u}_4 = 2+1-1-2=0$

the basis is orthogonal to be orthonormal convert to length one.

$\|u_1\| = \sqrt{1+4+1+1} = \sqrt{7}$

$\|u_2\| = \sqrt{4+1+1+1} = \sqrt{7}$

$\|u_3\| = \sqrt{1+1+4+1} = \sqrt{7}$

$\|u_4\| = \sqrt{1+1+1+4} = \sqrt{7}$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1/\sqrt{7} \\ 2/\sqrt{7} \\ 1/\sqrt{7} \\ 1/\sqrt{7} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{7} \\ 1/\sqrt{7} \\ -1/\sqrt{7} \\ 1/\sqrt{7} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{7} \\ 1/\sqrt{7} \\ -2/\sqrt{7} \\ -1/\sqrt{7} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{7} \\ 1/\sqrt{7} \\ 1/\sqrt{7} \\ -2/\sqrt{7} \end{bmatrix} \right\}$$

$$\vec{X}_\perp = \text{proj}_{u_4} \vec{X} = 4(-1/\sqrt{7}) + 5/\sqrt{7} - 3/\sqrt{7} + 3(-2/\sqrt{7}) = \frac{-4+5-3-6}{\sqrt{7}} = \frac{-8}{\sqrt{7}} \vec{u}_4$$

$$\vec{X}_\parallel = \vec{X} - \vec{X}_\perp =$$

$$\begin{bmatrix} 8/7 \\ -8/7 \\ -8/7 \\ 16/7 \end{bmatrix} = X_\perp$$

$$\begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix} - \begin{bmatrix} 8/7 \\ -8/7 \\ -8/7 \\ 16/7 \end{bmatrix} = \begin{bmatrix} 20/7 \\ 43/7 \\ -13/7 \\ 5/7 \end{bmatrix} = \vec{X}_\parallel$$

(fast way, since $W^\perp$ has fewer vectors than $W$ & $W^\perp$ is known)

b) $\vec{u}_1 \cdot \vec{u}_2 = -1+1+0=0$

normalize: $\|u_1\| = \sqrt{2}$ $\quad \|u_2\| = \sqrt{2}$

76 cont. $\hat{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ $\hat{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

$\vec{X}_{\parallel} = X \cdot \hat{u}_1 [u_1] + X \cdot \hat{u}_2 [u_2]$ since $u_i \cdot u_i = 1$ for normalized vectors

$= (-1)(1/\sqrt{2}) + 4(1/\sqrt{2}) + 0 = 3/\sqrt{2}$ $\quad -1(-1/\sqrt{2}) + 4(1/\sqrt{2}) = 5/\sqrt{2}$

$\frac{3}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} + \frac{5}{\sqrt{2}} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5/2 \\ 5/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$ $\quad X_{\perp} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

c) $\vec{u}_1 \cdot u_2 = 1 + 0 + 0 - 1 = 0$ $\quad \|u_1\| = \sqrt{3}$

$\vec{u}_1 \cdot \vec{u}_3 = 0 - 1 + 0 + 1 = 0$ $\quad \|u_2\| = \sqrt{3}$

$\vec{u}_2 \cdot \vec{u}_3 = 0 + 0 + 1 - 1 = 0$ $\quad \|u_3\| = \sqrt{3}$

$B = \left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \\ -1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} \right\}$

$\vec{X}_{\parallel} = \text{proj}_W \vec{X} =$

$3/\sqrt{3} + 4/\sqrt{3} + 0 - 6/\sqrt{3} = 1/\sqrt{3}$

$3/\sqrt{3} + 0 + 5/\sqrt{3} + 6/\sqrt{3} = 14/\sqrt{3}$

$0 - 4/\sqrt{3} + 5/\sqrt{3} - 6/\sqrt{3} = -5/\sqrt{3}$

$1/\sqrt{3}[u_1] + 14/\sqrt{3}[u_2] + -5/\sqrt{3}[u_3] =$

$\begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ -1/3 \end{bmatrix} + \begin{bmatrix} 14/3 \\ 0 \\ 14/3 \\ 14/3 \end{bmatrix} + \begin{bmatrix} 0 \\ +5/3 \\ -5/3 \\ +5/3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} = \vec{X}_{\parallel}$

$\vec{X}_{\perp} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$

the best approximation to $\vec{X}$ in $W$ is $\vec{X}_{\parallel}$.

the distance is the length of $\vec{X}_{\perp}$

a) $\sqrt{\frac{64}{49} + \frac{64}{49} + \frac{64}{49} + \frac{256}{49}} = \sqrt{\frac{384}{49}} = \frac{8\sqrt{6}}{7}$

b) 3

c) $\sqrt{4+4+4+0} = \sqrt{12} = 2\sqrt{3}$

8. a) $A$ is not linearly independent, so $(A^TA)^{-1}A^T\vec{b}=\vec{x}$ will not work ⑩

but there may be a solution if we row reduce $A^TA\vec{x}=A^T\vec{b}$

$$A^T=\begin{bmatrix}1&1&1&1\\1&1&0&0\\0&0&1&1\end{bmatrix}\qquad A^TA=\begin{bmatrix}4&2&2\\2&2&0\\2&0&2\end{bmatrix}\qquad A^T\vec{b}=\begin{bmatrix}14\\4\\10\end{bmatrix}$$

$$A^TA\vec{x}=A^T\vec{b}\Rightarrow\left[\begin{array}{ccc|c}4&2&2&14\\2&2&0&4\\2&0&2&10\end{array}\right]\text{ rref}\Rightarrow$$

$$\left[\begin{array}{ccc|c}1&0&1&5\\0&1&-1&-3\\0&0&0&0\end{array}\right]\qquad \begin{aligned}x_1+x_3&=5\\x_2-x_3&=-3\\x_3&=x_3\end{aligned}\Rightarrow\vec{x}=\begin{bmatrix}-1\\1\\1\end{bmatrix}x_3+\begin{bmatrix}5\\-3\\0\end{bmatrix}$$

infinite # of solutions choose $x_3$.

b) $(A^TA)^{-1}A^T\vec{b}=\vec{x}=\begin{bmatrix}-4\\3\end{bmatrix}$

9a)
$$\begin{aligned}0&=\beta_0+\beta_1(1)\\1&=\beta_0+2\beta_1\\2&=\beta_0+4\beta_1\\3&=\beta_0+5\beta_1\end{aligned}\qquad A=\begin{bmatrix}1&1\\1&2\\1&4\\1&5\end{bmatrix}\qquad\vec{b}=\begin{bmatrix}0\\1\\2\\3\end{bmatrix}$$

$$(A^TA)^{-1}A^T\vec{b}=\begin{bmatrix}-.6\\.7\end{bmatrix}$$

b)
$$\begin{aligned}0&=\beta_0+\beta_1+\beta_2\\1&=\beta_0+2\beta_1+4\beta_2\\2&=\beta_0+4\beta_1+16\beta_2\\3&=\beta_0+5\beta_1+25\beta_2\end{aligned}\qquad A=\begin{bmatrix}1&1&1\\1&2&4\\1&4&16\\1&5&25\end{bmatrix}\qquad\vec{b}=\begin{bmatrix}0\\1\\2\\3\end{bmatrix}$$

$$(A^TA)^{-1}A^T\vec{b}=\begin{bmatrix}-.6\\.7\\6.8\times10^{-3}\end{bmatrix}\Rightarrow\begin{bmatrix}-.6\\.7\\0\end{bmatrix}$$

a quadratic is not a good approximation for this data since the coefficient of $x_2$ is essentially zero.

9c)

$$1.58 = 4\beta_1 + 16\beta_2 + 64\beta_3$$
$$2.08 = 6\beta_1 + 36\beta_2 + 216\beta_3$$
$$2.5 = 8\beta_1 + 64\beta_2 + 512\beta_3$$
$$2.8 = 10\beta_1 + 100\beta_2 + 1000\beta_3$$
$$3.1 = 12\beta_1 + 144\beta_2 + 1728\beta_3$$
$$3.4 = 14\beta_1 + 196\beta_2 + 2744\beta_3$$
$$3.8 = 16\beta_1 + 256\beta_2 + 4096\beta_3$$
$$4.32 = 18\beta_1 + 324\beta_2 + 5832\beta_3$$

$$A = \begin{bmatrix} 4 & 16 & 64 \\ 6 & 36 & 216 \\ 8 & 64 & 512 \\ 10 & 100 & 1000 \\ 12 & 144 & 1728 \\ 14 & 196 & 2744 \\ 16 & 256 & 4096 \\ 18 & 324 & 5832 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1.58 \\ 2.08 \\ 2.5 \\ 2.8 \\ 3.1 \\ 3.4 \\ 3.8 \\ 4.32 \end{bmatrix}$$

$$(A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} .5132\ldots \\ -.0334\ldots \\ .00101\ldots \end{bmatrix}$$