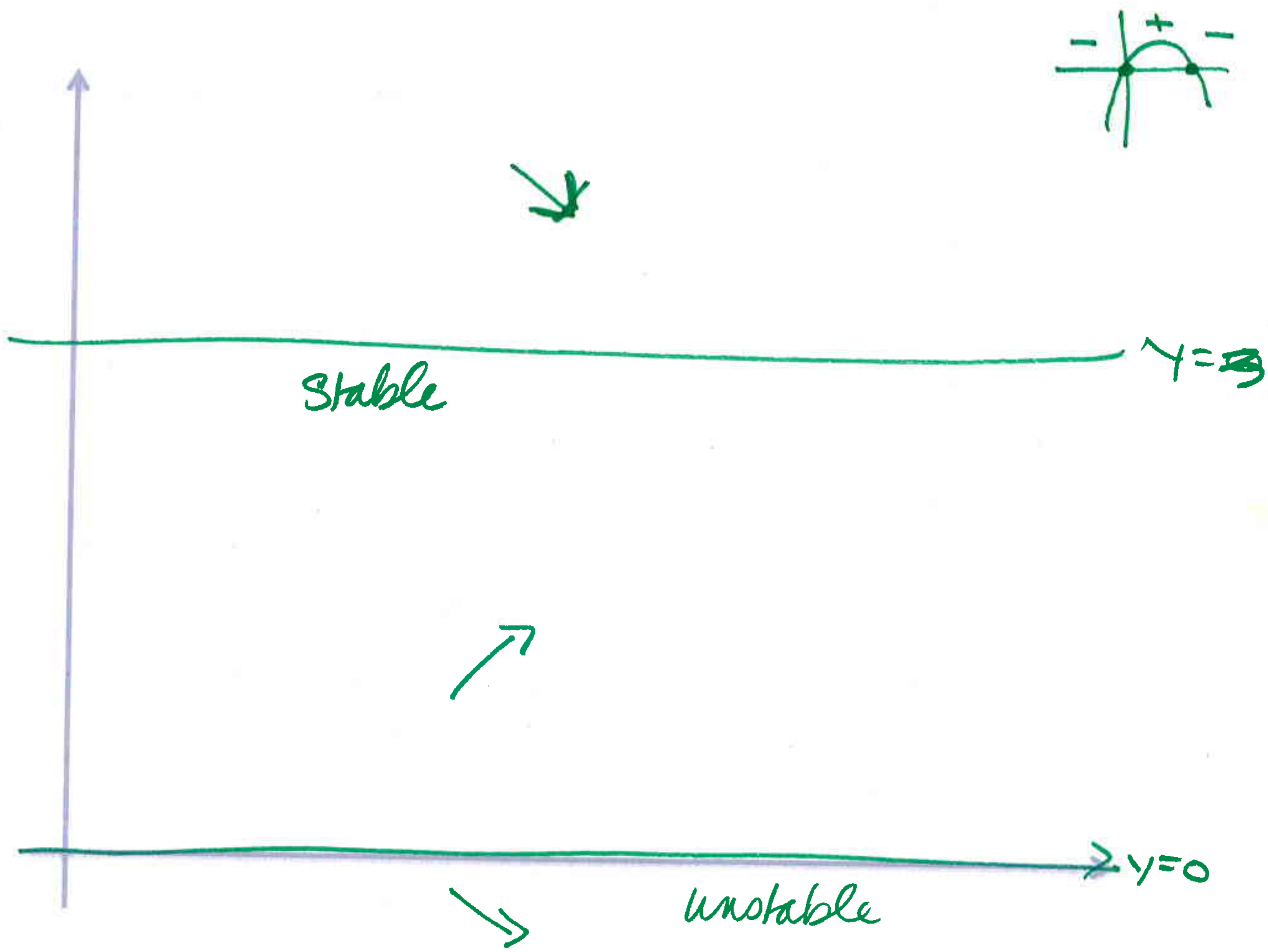
Name KEY

Math 285, Exam #1, Spring 2012

Instructions: Show all work. Use exact answers whenever possible (word problems being the only exception). Be sure to answer all parts of each question.

1. Consider the differential equation $y' = y(3-y)$. Determine the equilibrium (stationary) points. Label each equilibrium value of the graph below and sketch the direction field. You may draw the phase plane rather than plotting specific points to obtain the general behaviour. Determine whether any equilibrium points are asymptotically stable or unstable. (20 points)



2. Classify the following differential equations as linear or nonlinear, ordinary or partial, and their order. (3 points each)

a. $\frac{d^2y}{dt^2} + \sin(t+y) = \sin t$ 2nd order, nonlinear, ordinary

b. $\frac{\partial^2 y}{\partial y \partial x} + 4\frac{\partial^3 y}{\partial x^3} = 0$ partial, 3rd order, linear

c. $y' + (\sin t)y = y^{-2}$ 1st order, ordinary, nonlinear

d. $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = u_t$ 4th order, linear, partial

3. Determine if the proposed solution $y(t) = 3t + t^2$ satisfies the differential equation $ty' - y = t^2$. (8 points)

$y' = 3 + 2t$

$y = 3t + t^2$

$t(3+2t) - (3t + t^2) = \cancel{3t} + 2t^2 - \cancel{3t} - t^2 = t^2$ ✓

4. Determine which method should be used to solve the following equations. Choose from exact, linear (give the integrating factor), separable, homogeneous (state the degree), Bernoulli (state n). Do not solve the equations, just determine the method to be used. (5 points each)

a. $(1+t^2)y' + 8ty = (1+t^2)^{-2}$

linear

$y' + \frac{8t}{1+t^2} = (1+t^2)^{-3}$

$\mu = e^{\int \frac{8t}{1+t^2} dt} = e^{4\ln|1+t^2|} = \boxed{(1+t^2)^4}$

b. $y' - 7x^2y = x^2y^5$

Bernoulli $n = 5$

c. $y' = \frac{x^2+y^2}{2xy}$ homogeneous degree 0

d. $(e^x \sin y + 2y)dx - (3x - e^x \cos y)dy = 0$ exact? may require integrating factor or more

$M_y = e^x \cos y + 2$ $N_x = e^x \cos y - 3$

e. $\sin x \, dx + \cos 2y \, dy = 0$ Separable

5. Solve the differential equation and find the general solution $y' + 2y = te^{-2t}$ for the initial condition y(1)=0. (25 points)

$$\mu = e^{\int 2\,dt} = e^{2t}$$

$$e^{2t}y' + 2e^{2t}y = te^{-2t}e^{2t}$$

$$\int (e^{2t}y)' = \int t$$

$$e^{2t}y = \frac{t^2}{2} + C$$

$$\boxed{y = \frac{1}{2}t^2e^{-2t} + Ce^{-2t}}$$

$$0 = \frac{1}{2}(1)\cancel{e^{-2}} + C\cancel{e^{-2}}$$

$$C = -\frac{1}{2}$$

$$y = \frac{1}{2}t^2e^{-2t} - \frac{1}{2}e^{-2t}$$

$$y = \frac{1}{2}e^{-2t}(t^2-1)$$

6. Verify that the differential equation $y' = \frac{3y^2 - x^2}{2xy}$ is homogeneous. Then solve the equation by that method. (25 points)

$tx \Rightarrow x$

$ty \Rightarrow y$

$$\frac{3t^2y^2 - t^2x^2}{2t^2xy} = \frac{\cancel{t^2}}{\cancel{t^2}}\left(\frac{3y^2-x^2}{2xy}\right) \quad t^0$$

degree 0 homogeneous

$y = vx \qquad y' = v'x + v$

$$v'x + \cancel{v} = \frac{3v^2\cancel{x^2} - \cancel{x^2}}{2\cancel{x^2}v} = \frac{3v^2-1}{2v} - v\,\frac{2v}{2v}$$

$$v'x = \frac{3v^2-1-2v^2}{2v} = \frac{v^2-1}{2v}$$

$$\frac{dv \cdot 2v}{v^2-1} = \frac{1}{x}\,dx \qquad v = y/x$$

$$\int \frac{2v}{v^2-1}\,dv = \int \frac{1}{x}\,dx$$

$$\ln|v^2-1| = \ln x + C$$

$$v^2 - 1 = Ax$$

$$v^2 = Ax + 1$$

$$v = \pm\sqrt{Ax+1} \rightarrow \frac{y}{x} = \pm\sqrt{Ax+1}$$

$$\boxed{y = \pm x\sqrt{Ax+1}}$$

7. A pool contains 40,000 gallons of pure water. A pool cleaner wishes to add chlorine to the water at the rate of 500 ppm/gallon in water added at 5 gallons/minute. He achieves this by pumping water out of the pool at the same rate. Assuming that the chlorine is well-mixed in the pool, write a differential equation that represents the amount of chlorine in the pool in ppm (parts-per-million) at any given time t. Solve the equation and then determine how long it will take to get the entire pool to 15 ppm. (25 points)

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

$$\frac{dQ}{dt} = \frac{500\,ppm}{gal} \cdot \frac{5\,gal}{min} - \frac{5\,gal}{min} \cdot \frac{Q}{40000\,gal}$$

$$\frac{dQ}{dt} = 2500 - 1.25\times10^{-4}\,Q =$$

$$-1.25\times10^{-4}\,(-2\times10^{7} + Q)$$

$Q(0)=0$

$Q = 2\times10^{7} - A$

$A = -2\times10^{7}$

$$\int \frac{dQ}{Q-2\times10^{7}} = \int -1.25\times10^{-4}\,dt$$

$Q(t) = 15$

$$\ln|Q-2\times10^{7}| = -1.25\times10^{-4}t + C$$

$$Q - 2\times10^{7} = Ae^{-1.25\times10^{-4}t}$$

$$Q = 2\times10^{7} + Ae^{-1.25\times10^{-4}t}$$

$$Q = 2\times10^{7} - (2\times10^{7})e^{-1.25\times10^{-4}t}$$

$$\frac{15-2\times10^{7}}{-2\times10^{7}} = e^{-1.25\times10^{-4}t}$$

$$-7.5\times10^{-7} = -1.25\times10^{-4}t$$

$$\boxed{t = .006\ \text{min}}$$

8. Consider the nonlinear differential equation $y' = (1 - t^2 - y^2)^{1/2}$. Find the regions in the ty-plane where solutions will be certain to be continuous. (10 points)

$f(t,y) = (1-t^2-y^2)^{1/2}$

continuous where $1-t^2-y^2 \geq 0$

$1 \geq t^2+y^2$ inside unit circle

not sep. so $\int$ check does not apply

$\frac{\partial f}{\partial y} = \frac{1}{2}(1-t^2-y^2)^{-1/2} \cdot -2y = \frac{y}{\sqrt{1-t^2-y^2}} \Rightarrow 1-t^2-y^2 \neq 0$

defined strictly inside unit circle

& not the boundary.

$\boxed{t^2+y^2<1}$

9. Determine if the equation $(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0$ is exact. If it is, find ~~ψ(x).~~ (15 points)

M N

$\psi(x,y)$

$M_y = \cos x + 2xe^y$

✓

$N_x = \cos x + 2xe^y$

$\int y\cos x + 2xe^y \, dx = y\sin x + x^2e^y + f(y)$

$\int \sin x + x^2e^y - 1 \, dy = y\sin x + x^2e^y - y + g(x)$

$\boxed{\psi(x,y) = y\sin x + x^2e^y - y}$