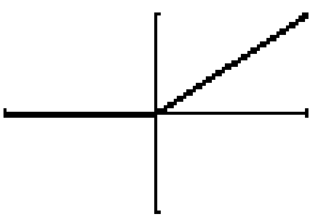
Sketching Fourier Series

Fourier series are defined on functions defined on an interval symmetric to the origin (-L,L), and which are periodic since the series we are representing these functions with, sine and cosine, are also periodic. As long as we are interested only in a finite interval, however, we don't care what happens outside the (-L,L) interval.

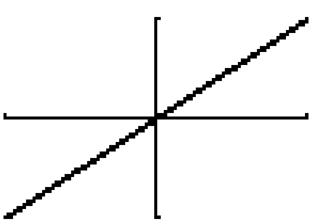
More problematic are functions defined on only a portion of the interval we need, say, only (0,L). In such a case we have a number of options available to us to extend the function to the required interval so that we can go ahead and calculate the Fourier series.

Example 1. It would at first appear that the simplest way to do this is to set the missing component to zero. Consider the function $f(x) = x, 0 < x < 1$. If we define the function to be zero on the interval $-1 < x \leq 0$, our function becomes: $f(x) = \begin{cases} 0, & -1 < x \leq 0 \\ x, & 0 < x < 1 \end{cases}$. And the Fourier integral, for the a_m coefficient becomes $a_m = \int_{-L}^{L} f(x)\cos(mx)\,dx = \int_{-1}^{0} 0\cos(mx)\,dx + \int_0^1 x cos(mx)dx = \int_0^1 x cos(mx)dx$. Which is just what you'd expect from ignoring the fact that the interval isn't symmetric. The graph of our periodic piece would look like this:

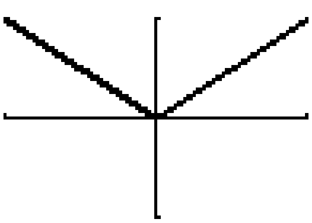


The difficulty here is that the graph isn't symmetric. And so while it at first appears that the integration will be easier, none of our sine or cosine coefficients will go to zero, and so our Fourier series will be very complicated.

Another way around this problem is to extend the function in a way that creates some kind of symmetry (either even symmetry: y-axis symmetry; or odd symmetry: origin symmetry), since the sine function is odd and the cosine function is even, you don't need odd functions to approximate an even function, and you don't need even functions to approximate an odd function. One of the coefficients will be zero, and we won't need to integrate all the sine and cosine integrals every time, plus our series will appear more compact, and still converge on the half of the interval we'd started with.



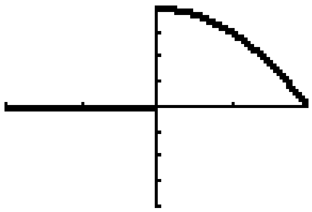
Since the graph of f(x) = x is already odd, extending this function to be odd, and thus contain sine only terms, just involves extending the interval. So $f(x) = x, 0 < x < 1$ becomes $f(x) = x, -1 < x < 1$. Then by symmetry, we can call $a_m = 0$ and only have to solve for b_m.



We can do something similar if we want to represent this as only cosine functions. Cosines are even, so we need to extend our f(x) from the original odd function, to be symmetric to it on the other side of the y-axis. We can change the symmetry by putting a negative sign through the function.

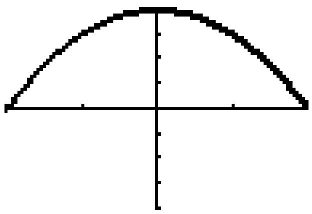
We now define the function as $f(x) = \begin{cases} -x, & -1 < x \leq 0 \\ x, & 0 < x < 1 \end{cases}$.

Example 2. Extend the function $f(x) = 4 - x^2, 0 < x < 2$, the same way we did in Example 1, as a function with no symmetry, as a function with odd symmetry, and as a function with even symmetry.



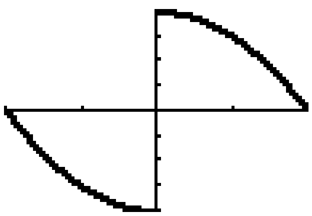
To extend it as a function with no symmetry, we can extend it any way we like to make the interval (-2,2), but we typically choose 0 in this case since it doesn't add any new integration complications. So, as in Example 1, we just define the function piecewise, with the left side of the interval equal to zero.

$$f(x) = \begin{cases} 0, & -2 < x \le 0 \\ 4 - x^2, & 0 < x < 2 \end{cases}$$



It's easier to make this function even than it is to make it odd, since the function is already even. As we did with the odd function in Example 1, all we have to do to extend this function as even is to extend the interval over which it's defined. $f(x) = 4 - x^2, -2 < x < 2$

Making it odd, requires flipping the sign of the function on the extension.



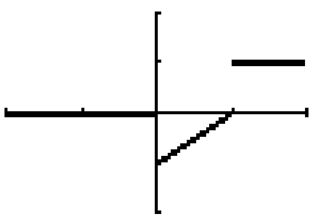
$$f(x) = \begin{cases} x^2 - 4, & -2 < x \le 0 \\ 4 - x^2, & 0 < x < 2 \end{cases}$$

The symmetry is readily apparent from the graph, and so it can be really helpful to know what to do by graphing what you need the function to look like first.

Now, let's look at a case with multiple pieces.

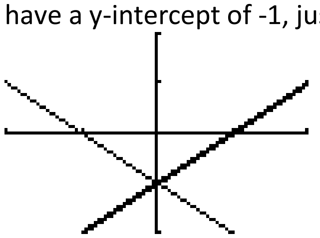
Example 3. Extend the function $f(x) = \begin{cases} x - 1, & 0 < x \le 1 \\ 1, & 1 < x < 2 \end{cases}$, the same way we did in Example 1, as a function with no symmetry, as a function with odd symmetry, and as a function with even symmetry.

This will be a little trickier for the even and odd functions, but if we don't care about the symmetry, nothing has changed. The new function just adds the 0 to extend the missing interval:



$$f(x) = \begin{cases} 0, & -2 < x \le 0 \\ x - 1, & 0 < x \le 1 \\ 1, & 1 < x < 2 \end{cases}$$

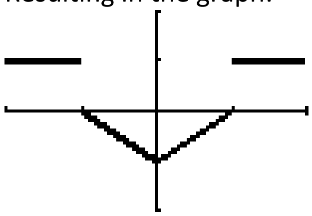
To extend the function as an even interval, the piece where f(x)=1 is already even, so that we just have to map unto the interval -2<x<-1. The piece of x-1 is going to be a bit trickier. To be even, it will need to have a y-intercept of -1, just as this line does, but the slope will need to go in the opposite direction.



Notice on the graph to the left how the line y= -x-1 has the appropriate profile. Because we want even symmetry we switched the sign of the odd component of the function and left the even component, the constant, the same. This gives us the desired result.

$$f(x) = \begin{cases} 1, & -2 < x \le -1 \\ -x-1, & -1 < x \le 0 \\ x-1, & 0 < x \le 1 \\ 1, & 1 < x < 2 \end{cases}$$

Resulting in the graph:

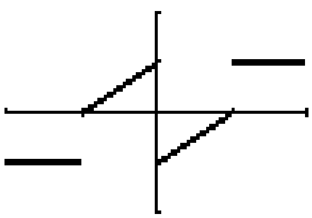


Oh, happy day. ☺

To extend the function as odd, we will do a similar trick. Any even components need to have their signs changed, but odd components can stay the same. This would result in the function:

$$f(x) = \begin{cases} -1, & -2 < x \le -1 \\ x+1, & -1 < x \le 0 \\ x-1, & 0 < x \le 1 \\ 1, & 1 < x < 2 \end{cases}$$

We can check the graph to see if this worked:



And it does. Odd symmetry has the 2nd/4th quadrants as mirror images, and the 1st/3rd as mirror images.

Practice Problems.

Extend the functions below to obtain a) 0 on the interval (-L,0), b) a cosine series, c) a sine series. Sketch the graph of each.

1. $f(x) = \begin{cases} x, 0 < x < 2 \\ 1, 2 \le x < 3 \end{cases}$
2. $f(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi \le x < 2\pi \\ 2, & 2\pi \le x < 3\pi \end{cases}$
3. $f(x) = x - 3, 0 < x < 4$
4. $f(x) = x^2 - 2x, 0 < x < 2$
5. $f(x) = x^3 - 5x^2 + 5x + 1, 0 < x < 3$
6. $f(x) = \begin{cases} -1, & 0 < x < 2 \\ 1, & 2 \le x < 4 \\ 2x - 3, & 4 \le x < 6 \end{cases}$