Math 2255, Exam #2, Spring 2012 Name KEY

Instructions: Show all work. Use exact answers unless specifically asked to round.

1. Solve the following second order homogeneous linear differential equations for the general solution. (10 points each)

a. $y'' - 4y' - 12y = 0$

$r^2 - 4r - 12 = 0$

$(r-6)(r+2) = 0$

$r = 6, r = -2$

$\boxed{y_c(t) = c_1 e^{6t} + c_2 e^{-2t}}$

b. $y'' + 4y' + 13y = 0$

$r^2 + 4r + 13 = 0$

$r = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$

$\boxed{y_c(t) = c_1 e^{-2t} \cos 3t + c_2 e^{-2t} \sin 3t}$

c. $y'' + 10y + 25y = 0$

$r^2 - 10r + 25 = 0$

$(r+5)^2 = 0$

$r = -5$ repeated

$\boxed{y_c(t) = c_1 e^{-5t} + c_2 t e^{-5t}}$

d. $t^2y'' - 3ty' - 12y = 0$

$n^2 - n - 3n - 12 = n^2 - 4n - 12 = 0$

$(n-6)(n+2) = 0$

$n = 6,\ n = -2$

$\boxed{y_c(t) = c_1 t^6 + c_2 t^{-2}}$

e. $t^2y'' - 3ty' + 4y = 0$

$n^2 - n - 3n + 4 = 0$

$n^2 - 4n + 4 = 0$

$(n-2)^2 = 0$

$n = 2$ repeated

$\boxed{y_c(t) = c_1 t^2 + c_2 t^2 \ln t}$

2. For your answer in problem 1e, calculate the value of the Wronskian using the solutions obtained. Then verify your results using Abel's theorem. (15 points)

$$W = \begin{vmatrix} t^2 & t^2 \ln t \\ 2t & 2t\ln t + t \end{vmatrix} = \cancel{2t^3 \ln t} + t^3 - \cancel{2t^3 \ln t} = \boxed{t^3}$$

by Abel's theorem

$$y'' - \frac{3}{t}y' + \frac{4}{t^2}y = 0$$

$$W = c\,e^{-\int p(t)\,dt} = c \cdot e^{-\int -\frac{3}{t}dt} = c \cdot e^{\int \frac{3}{t}dt} = c \cdot e^{3\ln t} = c\,e^{\ln t^3}$$

$$= c \cdot t^3$$

3. Use the method of undetermined coefficients and your solution from 1c to find the particular solution to the nonhomogeneous differential equation $y'' + 10y + 25y = 4e^{-5t} + 7t$. (15 points)

$y_1 = e^{-5t} \qquad y_2 = te^{-5t}$

$Y(t) = At^2e^{-5t} + Bt + C \qquad Y'(t) = 2Ate^{-5t} - 5At^2e^{-5t} + B$

$Y''(t) = 2Ae^{-5t} - 10Ate^{-5t} - 10Ate^{-5t} + 25At^2e^{-5t}$

$2Ae^{-5t} \cancel{+20Ate^{-5t}} \cancel{+25t^2e^{-5t}} + \cancel{20Ate^{-5t}} \cancel{-50At^2e^{-5t}} - 10B$

$\cancel{+25At^2e^{-5t}} + 25Bt + 25C = 4e^{-5t} + 7t$

$2A = 4 \Rightarrow A = 2$

$25B = 7 \Rightarrow B = 7/25$

$-10B + 25C = 0$

$25C = \overset{2}{\cancel{10}}(7/\underset{5}{\cancel{25}}) = \frac{14}{5} \Rightarrow C = \frac{14}{125}$

$$\boxed{Y_p(t) = 2t^2e^{-5t} + 7/25\,t + \frac{14}{125}}$$

4. Use the method of variation of parameters and your solution from problem 1a to find the particular solution to $y'' - 4y' - 12y = 0$. (15 points)

te^{-2t}

$y_1 = e^{6t} \qquad y_2 = e^{-2t}$

$W = \begin{vmatrix} e^{6t} & e^{-2t} \\ 6e^{6t} & -2e^{-2t} \end{vmatrix} = -2e^{4t} - 6e^{4t} = -8e^{4t}$

$y(t) = -e^{6t}\int \frac{te^{-2t} \cdot e^{-2t}}{-8e^{4t}}\,dt + e^{-2t}\int \frac{te^{-2t}e^{6t}}{-8e^{4t}}\,dt$

$= -e^{6t}\int te^{-8t}\,dt + e^{-2t}\int t\,dt =$

u	dv
t	e^{-8t}
1	$-\frac{1}{8}e^{-8t}$
	$\frac{1}{64}e^{-8t}$

$= -e^{6t}\left[-\frac{1}{8}te^{-8t} - \frac{1}{64}e^{-8t} + c_1\right] +$

$e^{-2t}\left[t^2/2 + c_2\right] =$

$$\boxed{\frac{1}{8}te^{-2t} + \frac{1}{64}e^{-2t} + \frac{1}{2}t^2e^{-2t} + c_1e^{6t} + c_2e^{-2t}}$$

5. Use reduction of order to solve the differential equation $t^2y'' + 7ty' - 7y = 0$ given that $y_1 = t$. (10 points)

$y = vt \quad y' = v't + v \quad y'' = v''t + 2v'$

$t^2(v''t + 2v') + 7t(v't + v) - 7vt =$

$t^3v'' + 2t^2v' + 7t^2v' + \cancel{7vt} - \cancel{7vt} = 0 \qquad v' = u \quad v'' = u'$

$\dfrac{t^3v'' + 9t^2v' = 0}{t^3} \Rightarrow u' + \frac{9}{t}u = 0 \Rightarrow \int \frac{u'}{u} = \int -\frac{9}{t}$

$\ln u = -9\ln t \Rightarrow u = t^{-9} = v' \Rightarrow v = \int t^{-9}\,dt = \dfrac{t^{-8}}{-8}$

$y(t) = \left(-\frac{1}{8}t^{-8} + c_1\right)t \Rightarrow \boxed{y(t) = c_2 t^{-7} + c_1 t}$

Check

$n^2 - n + 7n - 7 = 0 \Rightarrow n^2 + 6n - 7 = 0 \quad (n+7)(n-1) = 0 \Rightarrow n = -7, n = 1$

$y_1 = t^{-7}, \; y_2 = t$ ✓

6. Solve the higher order homogeneous differential equations for the general solutions. (10 points each)

a. $y''' - y'' - y' + y = 0$

$r^3 - r^2 - r + 1 = 0$

$r^2(r-1) - 1(r-1) = 0$

$(r^2-1)(r-1) = 0 \Rightarrow (r-1)(r-1)(r+1) = 0 \qquad r = 1$, repeated

$r = -1$

$\boxed{y_c(t) = c_1 e^t + c_2 t e^t + c_3 e^{-t}}$

b. $y^V - 5y''' + 4y' = 0$

$r^5 - 5r^3 + 4r = 0 \Rightarrow r(r^4 - 5r^2 + 4) \Rightarrow r = 0$

$r(r^2-4)(r^2-1) \Rightarrow r(r-2)(r+2)(r-1)(r+1)$

$r = 0, 1, 2, -1, -2$

$y_c(t) = c_1 + c_2 e^t + c_3 e^{2t} + c_4 e^{-t} + c_5 e^{-2t}$

7. Set up the Wronskian matrix, but do not calculate it, for the problem in 6b. What does Abel's theorem tell you its value should be? (7 points)

$$W = \begin{vmatrix} 1 & e^t & e^{-t} & e^{2t} & e^{-2t} \\ 0 & e^t & -e^{-t} & 2e^{2t} & -2e^{-2t} \\ 0 & e^t & e^{-t} & 4e^{2t} & 4e^{-2t} \\ 0 & e^t & -e^{-t} & 6e^{2t} & -6e^{-2t} \\ 0 & e^t & e^{-t} & 8e^{2t} & 8e^{2t} \end{vmatrix}$$

$p(t)y''' \Rightarrow p(t) = 0$

$W = c e^{-\int 0 dt} = \boxed{c}$

8. Convert e^{1+2i} into a complex number of the form a+bi using Euler's formula. Give an exact answer in terms of exponentials and square roots. (6 points)

$e[e^{2i}] = \boxed{e[\cos(2) + i\sin(2)]}$ (2 is in radians)

$= e\cos 2 + ie\sin 2$

9. Convert 1-3i into a complex exponential in the form of $Re^{i\theta}$. Be sure that θ is in radians. You may round your answer for θ to 4 decimal places. (7 points)

$1^2 + (-3)^2 = 10$

$\|1-3i\| = \sqrt{10}$

$1+3i = \sqrt{10}\left(\frac{1}{\sqrt{10}} - \frac{3}{\sqrt{10}} i\right)$

$\theta = \tan^{-1}\left(\frac{-3}{1}\right) \Rightarrow \tan^{-1}(-3) \pm k\pi$

$\sqrt{10}\, e^{i\tan^{-1}(-3)} \approx \boxed{\sqrt{10}\, e^{-1.2490i}}$

10. Find all the 6th roots of -1. Write exact answers in the form a+bi. (8 points)

$\sqrt[6]{-1} \Rightarrow \left(e^{\pi i}\right)^{1/6} \Rightarrow e^{\frac{\pi}{6} i} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$\left(e^{3\pi i}\right)^{1/6} \Rightarrow e^{\frac{\pi}{2} i} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$\left(e^{5\pi i}\right)^{1/6} \Rightarrow e^{\frac{5\pi}{6} i} = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

$\left(e^{7\pi i}\right)^{1/6} \Rightarrow e^{\frac{7\pi}{6} i} = \cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

$\left(e^{9\pi i}\right)^{1/6} \Rightarrow e^{\frac{3\pi}{2} i} = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i$

$\left(e^{11\pi}\right)^{1/6} \Rightarrow e^{11\pi/6\, i} = \cos\left(11\pi/6\right) + i\sin\left(11\pi/6\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

$\Rightarrow \boxed{\frac{\sqrt{3}}{2} \pm \frac{1}{2}i,\ \pm i,\ -\frac{\sqrt{3}}{2} \pm 1/2\, i}$

11. Given your answer from 1d. find the solution to the initial value problem $t^2y'' - 3ty' - 12y = 0, y(1) = 1, y'(1) = 0$. (7 points)

$y_c(t) = c_1 t^6 + c_2 t^{-2}$

$y'(t) = 6c_1 t^5 - 2c_2 t^{-3}$

$1 = c_1 + c_2 \xrightarrow{\times 2}$

$0 = 6c_1 - 2\cancel{c_2}$

$2 = 2c_1 + \cancel{2}c_2$

$2 = 8c_1 \Rightarrow c_1 = 1/4$

$\Rightarrow c_2 = 3/4$

$\boxed{y(t) = 1/4\, t^6 + 3/4\, t^{-2}}$