Differential Equations Series Solutions Key

used Throughout

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

$$y''' = \sum_{n=3}^{\infty} a_n n(n-1)(n-2) x^{n-3}$$

(#n only)

a. $y'' - y = 0 \quad x_0 = 0$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) - a_n\right]x^n = 0$$

$$\frac{a_n}{(n+2)(n+1)} = a_{n+2}$$

evens

$$a_2 = \frac{a_0}{2\cdot 1}$$

$$a_4 = \frac{a_2}{4\cdot 3} = \frac{a_0}{4\cdot 3\cdot 2\cdot 1}$$

$$a_6 = \frac{a_4}{6\cdot 5} = \frac{a_0}{6!}$$

odds

$$a_3 = \frac{a_1}{3\cdot 2}$$

$$a_5 = \frac{a_3}{5\cdot 4} = \frac{a_1}{5\cdot 4\cdot 3\cdot 2}$$

$$a_7 = \frac{a_5}{7\cdot 6} = \frac{a_1}{7!}$$

evens odds

$$a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = a_0 \cosh x + a_1 \sinh x$$

b. $y'' + 4y = 0 \quad x_0 = 1 \quad y = \sum_{n=0}^{\infty} a_n (x-1)^n$

$$\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} + 4\sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n + \sum 4a_n(x-1)^n =$$

$$\sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) + 4a_n\right](x-1)^n = 0 \qquad a_{n+2} = \frac{-4a_n}{(n+2)(n+1)}$$

evens

$$a_2 = \frac{-4a_0}{2\cdot 1}$$

$$a_4 = \frac{-4a_2}{4\cdot 3} = \frac{(-4)^2 a_0}{4!}$$

$$a_6 = \frac{-4a_4}{6\cdot 5} = \frac{(-1)^3(2)^6 a_0}{6!}$$

odds

$$a_3 = \frac{-4a_1}{3\cdot 2}$$

$$a_5 = \frac{-4a_3}{5\cdot 4} = \frac{(-4)^2 a_1}{5!}$$

$$a_7 = \frac{-4a_5}{7\cdot 6} = \frac{(-1)^3 2^6 a_1}{7!}$$

$$a_0 \sum_{n=0}^{\infty} \frac{(-1)^n (2)^{2n} (x-1)^n}{(2n)!} + \frac{a_1}{2} \sum \frac{(-1)^n (2)^{2n+1} (x-1)^{2n+1}}{(2n+1)!}$$

b. cont'd.

$$y = a_0 \cos 2(x-1) + \frac{a_1}{2} \sin 2(x-1)$$

c. $y'' + 4y' + 4y = 0 \quad x_0 = 0$

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + 4\sum_{n=1}^{\infty} a_n n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) x^n + \sum_{n=0}^{\infty} 4a_{n+1}(n+1) x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) + 4a_{n+1}(n+1) + 4a_n \right] x^n = 0$$

$$a_{n+2} = \frac{-4a_n - 4a_{n+1}(n+1)}{(n+2)(n+1)} = \frac{-4a_n}{(n+2)(n+1)} - \frac{4a_{n+1}}{(n+2)}$$

$n=0$ $\quad a_2 = \frac{-4a_0}{2\cdot 1} - \frac{4a_1}{2} = -2a_0 - 2a_1$

$n=1$ $\quad a_3 = \frac{-4a_1}{3\cdot 2} - \frac{4a_2}{3} = \frac{-4a_1}{6} - \frac{4}{3}\left(\frac{-4a_0}{2} - \frac{4a_1}{2}\right) = -\frac{4a_1}{6} + \frac{16a_0}{6} + \frac{16a_1}{6}$

$$= \frac{12a_1}{6} + \frac{16a_0}{6} = 2a_1 + \frac{8}{3}a_0$$

$n=2$ $\quad a_4 = \frac{-\cancel{4}a_2}{\cancel{4}\cdot 3} - \frac{\cancel{4}a_3}{\cancel{4}} = \frac{-a_2}{3} - a_3 = \frac{-1}{3}(2a_0 - 2a_1) - (2a_1 + \frac{8}{3}a_0) =$

$$-\frac{2}{3}a_0 + \frac{2}{3}a_1 - 2a_1 - \frac{8}{3}a_0 = \frac{-10}{3}a_0 - \frac{4}{3}a_1$$

$n=3$ $\quad a_5 = \frac{-\cancel{4}a_3}{5\cdot\cancel{4}} - \frac{4a_4}{5} = -\frac{1}{5}(2a_1 + \frac{8}{3}a_0) - \frac{4}{5}(-\frac{10}{3}a_0 - \frac{4}{3}a_1) =$

$$-\frac{2}{5}a_1 - \frac{8}{15}a_0 + \frac{8}{3}a_0 + \frac{16}{15}a_1 = \frac{32}{15}a_0 + \frac{2}{3}a_1$$

$$y = a_0\left(1 - 2x^2 + \frac{8}{3}x^3 - \frac{10}{3}x^4 + \frac{32}{15}x^5 + \dots\right) + a_1\left(x - 2x^2 + 2x^3 - \frac{4}{3}x^4 + \frac{2}{3}x^5 + \dots\right)$$

d. $y'' - y = 0 \quad x_0 = 3$

this will look exactly like a except replace x w/ $x-3$.
any values of a_0, a_1 will change w/ initial conditions but not the rest of the power series

e. $y'' + xy' + 2y = 0 \qquad x_0 = 0$ (3)

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} a_n n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$n=0$

$$a_2(2)(1)(x^0) + \sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} a_n n x^n + 2a_0 x^0 + \sum_{n=1}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=1}^{\infty}\left[a_{n+2}(n+2)(n+1) + a_n n + 2a_n\right]x^n + 2a_2 + 2a_0 = 0$$

$$a_{n+2} = \frac{-a_n\cancel{(n+2)}}{\cancel{(n+2)}(n+1)} = \frac{-a_n}{(n+1)}$$

$2a_2 = -2a_0$

$\Rightarrow a_2 = -a_0$

evens

$n=2$ $\quad a_4 = \frac{-a_2}{3} = \frac{-(-a_0)}{3} = \frac{a_0}{3}$

$n=4$ $\quad a_6 = \frac{-a_4}{5} = \frac{-(a_0/3)}{5} = -\frac{a_0}{15}$

$n=6$ $\quad a_8 = \frac{-a_6}{7} = \frac{a_0}{3\cdot5\cdot7} = \frac{a_0}{105}$

odd

$n=1$ $\quad a_3 = \frac{-a_1}{2}$

$n=3$ $\quad a_5 = \frac{-a_3}{4} = \frac{-(-a_1/2)}{4} = \frac{a_1}{8}$

$n=5$ $\quad a_7 = \frac{-a_5}{6} = -\frac{a_1}{2\cdot4\cdot6} = -\frac{a_1}{48}$

$$y = a_0\left(1 - x^2 + \tfrac{1}{3}x^4 - \tfrac{1}{15}x^6 + \tfrac{1}{105}x^8 + \dots\right) +$$
$$a_1\left(x - \tfrac{1}{2}x^3 + \tfrac{1}{8}x^5 - \tfrac{1}{48}x^7 + \dots\right)$$

f. $xy'' + y' + xy = 0 \qquad x_0 = 1$ (x=0 is a singular point)

$$(x-1)y'' + y'' + y' + (x-1)y + 1y = 0$$

$$(x-1)\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} + \sum_{n=2}^{\infty} a_n(n)(n-1)(x-1)^{n-2} + \sum_{n=1}^{\infty} a_n n(x-1)^{n-1}$$
$$+ (x-1)\sum_{n=0}^{\infty} a_n(x-1)^n + 1\sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\sum_{n=2}^{\infty} a_n(n)(n-1)(x-1)^{n-1} + \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} + \sum_{n=1}^{\infty} a_n n(x-1)^{n-1} +$$
$$\sum_{n=0}^{\infty} a_n(x-1)^{n+1} + \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

f cont'd

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^{n+1} + \sum_{n=-1}^{\infty} a_{n+3}(n+3)(n+2)(x-1)^{n+1} +$$

$$\sum_{n=-1}^{\infty} a_{n+2}(n+2)(x-1)^{n+1} + \sum_{n=0}^{\infty} a_n (x-1)^{n+1} + \sum_{n=-1}^{\infty} a_{n+1}(x-1)^{n+1} = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^{n+1} + a_2(2)(1)x^0 + \sum_{n=0}^{\infty} a_{n+3}(n+3)(n+2)(x-1)^{n+1} +$$

$$a_1(1)x^0 + \sum_{n=0}^{\infty} a_{n+2}(n+2)(x-1)^{n+1} + \sum_{n=0}^{\infty} a_n(x-1)^{n+1} + a_0(x^0) +$$

$$\sum_{n=0}^{\infty} a_{n+1}(x-1)^{n+1} = 0$$

$$2a_2 + a_1 + a_0 = 0 \implies 2a_2 = -a_0 - a_1 \implies \boxed{a_2 = -\tfrac{1}{2}a_0 - \tfrac{1}{2}a_1}$$

$$\sum_{n=0}^{\infty}\left[a_{n+2}(n+2)(n+1) + a_{n+3}(n+3)(n+2) + a_{n+2}(n+2) + a_n\right](x-1)^{n+1} = 0$$

$$a_{n+2}(n+2)\cancel{(n+1+1)} + a_{n+3}(n+3)(n+2) = -a_n - a_{n+2}(n+2)^2$$

$$a_{n+3} = \frac{-a_n}{(n+3)(n+2)} - \frac{a_{n+2}(n+2)^{\cancel{2}}}{(n+3)\cancel{(n+2)}}$$

$$a_{n+3} = \frac{-a_n}{(n+3)(n+2)} - a_{n+2}\frac{(n+2)}{(n+3)}$$

$n=0$

$$a_3 = \frac{-a_0}{(3)(2)} - a_2\left(\frac{2}{3}\right) = -\frac{a_0}{6} - \frac{2}{3}\left(-\frac{1}{2}a_0 - \frac{1}{2}a_1\right) = \frac{-a_0}{6} + \frac{1}{3}a_0 + \frac{1}{3}a_1$$

$$= \frac{1}{6}a_0 + \frac{1}{3}a_1$$

$n=1$

$$a_4 = \frac{-a_1}{4\cdot 3} - a_3\left(\frac{3}{4}\right) = -\frac{a_1}{12} - \frac{3}{4}\left(\frac{1}{6}a_0 + \frac{1}{3}a_1\right) = -\frac{1}{12}a_1 - \frac{1}{4}a_1 - \frac{1}{8}a_0 =$$

$$-\frac{5}{12}a_1 - \frac{1}{8}a_0$$

$n=2$

$$a_5 = -\frac{a_2}{5\cdot 4} - a_4\left(\frac{4}{5}\right) = \frac{-1}{20}\left(-\frac{1}{2}a_0 - \frac{1}{2}a_1\right) - \frac{4}{5}\left(-\frac{5}{12}a_1 - \frac{1}{8}a_0\right) =$$

$$\frac{1}{40}a_0 + \frac{1}{40}a_1 + \frac{1}{3}a_1 + \frac{1}{10}a_0 = \frac{1}{8}a_0 + \frac{43}{120}a_1$$

$$y = a_0\left(1 - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 - \tfrac{1}{8}x^4 + \tfrac{1}{8}x^5 + \cdots\right) + a_1\left(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{5}{12}x^4 + \tfrac{43}{120}x^5 + \cdots\right)$$

$g.\ x(x+3)^2y''-y=0.$ use $x_0=1$ since $x=0$ is a singular point

$$x(x^2+6x+9)=x^3+6x^2+9x$$

$$=(x-1)^3+9x^2+6x+1$$

$$=(x-1)^3+9(x-1)^2+24x-8$$

$$=(x-1)^3+9(x-1)^2+24(x-1)+16$$

$$(x-1)^3=x^3-3x^2+3x-1$$

$$x^3=(x-1)^3+3x^2-3x+1$$

$$(x-1)^2=x^2-2x+1$$

$$x^2=(x-1)^2+2x-1$$

$$6x+18x=24x$$

$$(x-1)=x-1\ +1 \qquad -8+24=16$$

$$(x-1)^3y''+9(x-1)^2y''+24(x-1)y''$$

$$+16y''-y=0$$

$$(x-1)^3\sum_{n=2}^{\infty}a_nn(n-1)(x-1)^{n-2}+9(x-1)^2\sum_{n=2}^{\infty}a_nn(n-1)(x-1)^{n-2}+24(x-1)\sum_{n=2}^{\infty}a_nn(n-1)(x-1)^{n-2}$$

$$+16\sum_{n=2}^{\infty}a_nn(n-1)(x-1)^{n-2}-\sum_{n=0}^{\infty}a_n(x-1)^n=0$$

$$\sum_{n=2}^{\infty}a_nn(n-1)(x-1)^{n+1}+\sum_{n=2}^{\infty}9a_nn(n-1)(x-1)^n+\sum_{n=2}^{\infty}24a_nn(n-1)(x-1)^{n-1}+\sum_{n=2}^{\infty}16a_nn(n-1)(x-1)^{n-2}$$

$$-\sum_{n=0}^{\infty}a_n(x-1)^n=0$$

$$\sum_{n=2}^{\infty}a_nn(n-1)(x-1)^{n+1}+\sum_{n=1}^{\infty}9a_{n+1}(n+1)n\,(x-1)^{n+1}+\sum_{n=0}^{\infty}24a_{n+2}(n+2)(n+1)(x-1)^{n+1}+$$

$$\sum_{n=-1}^{\infty}16\,a_{n+3}(n+3)(n+2)(x-1)^{n+1}-\sum_{n=-1}^{\infty}a_{n+1}(x-1)^{n+1}=0$$

$$\sum_{n=2}^{\infty}a_nn(n-1)(x-1)^{n+1}+9a_2(2)(1)(x-1)^2+\sum_{n=2}^{\infty}9a_{n+1}(n+1)(n)(x-1)^{n+1}+24a_2(2)(1)(x-1)^1$$

$$+24a_3(3)(2)(x-1)^2+\sum_{n=0}^{\infty}24a_{n+2}(n+2)(n+1)(x-1)^{n+1}+16a_2(2)(1)(x-1)^0+$$

$$16a_3(3)(2)(x-1)^1+16a_4(4)(3)(x-1)^2+\sum_{n=2}^{\infty}16a_{n+3}(n+3)(n+2)(x-1)^{n+1}-$$

$$a_0(x-1)^0-a_1(x-1)^1-a_2(x-1)^2-\sum_{n=2}^{\infty}a_{n+1}(x-1)^{n+1}=0$$

$$32a_2-a_0=0 \qquad 96a_3+48a_2-a_1=0 \qquad 18a_2+144a_3+192a_4-a_2=0$$

$$a_2=\tfrac{1}{32}a_0 \qquad a_3=\frac{a_1}{96}-\frac{48a_2}{96}=\frac{a_1}{96}-\frac{1}{2}\left(\frac{1}{32}a_0\right)=\frac{1}{96}a_1-\frac{1}{64}a_0$$

g cont'd

$$a_4 = \frac{-17}{192}a_2 - \frac{144}{192}a_3 = \frac{-17}{192}\left(\frac{1}{32}a_0\right) - \frac{3}{4}\left(\frac{1}{96}a_1 - \frac{1}{64}a_0\right) = \frac{55}{6144}a_0 - \frac{1}{128}a_1$$

$$\sum_{n=2}^{\infty}\Big[a_n n(n-1) + 9a_{n+1}(n+1)(n) + 24a_{n+2}(n+2)(n+1) + 16a_{n+3}(n+3)(n+2)$$
$$-a_{n+1}\Big](x-1)^{n+1} = 0$$

$$\cancel{16}a_{n+3}\cancel{(n+3)}\cancel{(n+2)} = \frac{-a_{n+2}\cdot \overset{3}{\cancel{24}}\cancel{(n+2)}(n+1)}{\underset{2}{\cancel{16}}(n+3)\cancel{(n+2)}} - \frac{a_{n+1}[9(n+1)(n)-1]}{16(n+3)(n+2)} - \frac{a_n n(n-1)}{16(n+3)(n+2)}$$

$$a_{n+3} = \frac{-a_{n+2}3(n+1)}{2(n+3)} - \frac{a_{n+1}[9(n+1)n-1]}{16(n+3)(n+2)} - \frac{a_n n(n-1)}{16(n+3)(n+2)}$$

$n=2$

$$a_5 = \frac{-a_4(3)(3)}{2(5)} - \frac{a_3\overset{53}{[9(3)(2)-1]}}{16(5)(4)} - \frac{a_2(\cancel{2})(1)}{16(5)(\cancel{4})2}$$

$$a_5 = -\frac{9}{10}\left(\frac{55}{6144}a_0 - \frac{1}{128}a_1\right) - \frac{53}{320}\left(\frac{1}{96}a_1 - \frac{1}{64}a_0\right) - \frac{1}{160}\left(\frac{1}{32}a_0\right)$$

$$= -\frac{33}{4096}a_0 + \frac{9}{1280}a_1 - \frac{53}{30720}a_1 + \frac{53}{20480}a_0 - \frac{1}{5120}a_0$$

$$-\frac{29}{5120}a_0 + \frac{163}{30720}a_1$$

$$y = a_0\left(1 + \frac{1}{32}x^2 - \frac{1}{64}x^3 + \frac{55}{6144}x^4 - \frac{29}{5120}x^5 + \cdots\right) +$$
$$a_1\left(x + \frac{1}{96}x^3 - \frac{1}{128}x^4 + \frac{163}{30720}x^5 + \cdots\right)$$

h. $y'' - \frac{1}{x}y' + \frac{1}{(x-1)^3}y = 0$ $0=x$ is a regular singular point

$1=x$ is an irregular singular point

$$x(x-1)^3y'' - (x-1)^3y' + xy = 0$$

$$(x^4-3x^3+3x^2-x)y'' - (x^3-3x^2+3x-1)y' + xy = 0$$

$$x^4y'' - 3x^3y'' + 3x^2y'' - xy'' - x^3y' + 3x^2y' - 3xy' + y' + xy = 0$$

n cont'd

assume the solution $y=\sum_{n=0}^{\infty} c_n x^{n+r}$

$$x^4\sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^{n+\cancel{r}-2} - 3x^3\sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^{n+\cancel{r}-2} + 3x^2\sum_{n=2} a_n(n+r)(n+r-1)x^{n+\cancel{r}}$$

$$-x\sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^{n+\cancel{r}-2} - x^3\sum_{n=1}^{\infty} a_n(n+r)x^{n+\cancel{r}-1} + 3x^2\sum_{n=1}^{\infty} a_n(n+r)x^{n+\cancel{r}-1}$$

$$-3x\sum_{n=1}^{\infty} a_n(n+r)x^{n+\cancel{r}-1} + \sum_{n=1}^{\infty} a_n(n+r)x^{n+\cancel{r}-1} + x\sum_{n=0} a_n x^{n+\cancel{r}} = 0$$

$$\sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^{n+2} - \sum_{n=2}^{\infty} 3a_n(n+r)(n+r-1)x^{n+1} + \sum_{n=2}^{\infty} 3a_n(n+r)(n+r-1)x^{n}$$

$$-\sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^{n-1} - \sum_{n=1}^{\infty} a_n(n+r)x^{n+2} + \sum_{n=1} 3a_n(n+r)x^{n+1} -$$

$$\sum_{n=1}^{\infty} 3a_n(n+r)x^{n} + \sum_{n=1}^{\infty} a_n(n+r)x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^{n+2} - \sum_{n=1}^{\infty} 3a_{n+1}(n+r+1)(n+r)x^{n+2} + \sum_{n=0}^{\infty} 3a_{n+2}(n+r+2)(n+r+1)x^{n+2}$$

$n=1$ $\quad$ $n=0, n=1$

$$-\sum_{n=-1}^{\infty} a_{n+3}(n+r+3)(n+r+2)x^{n+2} - \sum_{n=1}^{\infty} a_n(n+r)x^{n+2} + \sum_{n=0}^{\infty} 3a_{n+1}(n+r+1)x^{n+2} -$$

$n=-1,0,1$ $\quad$ $n=1$ $\quad$ $n=0,1$

$$\sum_{n=-1}^{\infty} 3a_{n+2}(n+r+2)x^{n+2} + \sum_{n=-2}^{\infty} a_{n+3}(n+r+3)x^{n+2} + \sum_{n=-1}^{\infty} a_{n+1}x^{n+2} = 0$$

$n=-1,0,1$ $\quad$ $n=-2,-1,0,1$ $\quad$ $n=-1,0,1$

$$\sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^{n+2} - 3a_2(r+2)(r+1)x^3 - \sum_{n=2}^{\infty} 3a_{n+1}(n+r+1)(n+r)x^{n+2}$$

$$+3a_2(r+2)(r+1)x^2 + 3a_3(r+3)(r+2)x^3 + \sum_{n=2}^{\infty} 3a_{n+2}(n+r+2)(n+r+1)x^{n+2}$$

$$-a_2(r+2)(r+1)x^1 - a_2(r+3)(r+2)x^2 - a_4(r+4)(r+3)x^3 - \sum_{n=2}^{\infty} a_{n+3}(n+r+3)(n+r+2)x^{n+2}$$

$$-a_1(r+1)x^3 - \sum_{n=2}^{\infty} a_n(n+r)x^{n+2} + 3a_1(r+1)x^2 + 3a_2(r+2)x^3 + \sum_{n=2}^{\infty} 3a_{n+1}(n+r+1)x^{n+2}$$

$$-3a_1(r+1)x^1 - 3a_2(r+2)x^2 - 3a_3(r+3)x^3 - \sum_{n=2}^{\infty} 3a_{n+2}(n+r+2)x^{n+2} + a_1(r+1)x^0$$

h cont'd

$$+a_2(r+2)x^1 + a_3(r+3)x^2 + a_4(r+4)x^3 + \sum_{n=2}^{\infty} a_{n+3}(n+r+3)x^{n+2}$$

$$+a_0 x + a_1 x^2 + a_2 x^3 + \sum_{n=2}^{\infty} a_{n+1} x^{n+2} = 0$$

$a_1(r+1)x^0 = 0 \implies a_1 = 0$ or $r+1=0$ choose $r=-1$

$$-a_2(r+2)(r+1)\cancel{x} - 3a_1(r+1)\cancel{x} + a_2(r+2)\cancel{x} + a_0\cancel{x} = 0$$

$$-a_2\cancel{(1)}(0) - 3\cancel{a_1}(0) + a_2(1) + a_0 = 0 \qquad \boxed{a_2 = -a_0}$$

$$3a_2(r+2)(r+1)\cancel{x^2} - a_2(r+3)(r+2)\cancel{x^2} + 3a_1(r+1)\cancel{x^2} - 3a_2(r+2)\cancel{x^2}$$

$$+a_3(r+3)\cancel{x^2} + a_1\cancel{x^2} = 0$$

$$\cancel{3a_2(1)(0)} - a_2(2)(1) - \cancel{3a_1(0)} - 3a_2(1) + a_3(2) + a_1 = 0$$

$$-2a_2 - 3a_2 + 2a_3 + a_1 = 0 \implies -5a_2 + 2a_3 + a_1 = 0$$

$$a_3 = \frac{5}{2}a_2 - \frac{1}{2}a_1 = \frac{5}{2}(-a_0) - \frac{1}{2}a_1 = \boxed{-\frac{5}{2}a_0 - \frac{1}{2}a_1}$$

$$-3a_2(r+2)(r+1)\cancel{x^3} + 3a_3(r+3)(r+2)\cancel{x^3} - a_4(r+4)(r+3)\cancel{x^3} - a_1(r+1)\cancel{x^3}$$

$$+3a_2(r+2)\cancel{x^3} - 3a_3(r+3)\cancel{x^3} + a_4(r+4)\cancel{x^3} + a_2\cancel{x^3} = 0$$

$$-\cancel{3a_2(1)(0)} + 3a_3(2)(1) - a_4(3)(2) - \cancel{a_1(0)} + 3a_2(1) - 3a_3(2)$$

$$+a_4(3) + a_2 = 0$$

$$6\cancel{a_3} - 6a_4 + 3a_2 - \cancel{6a_3} + 3a_4 + a_2 = 0$$

$$-3a_4 + 4a_2 = 0 \implies \boxed{a_4 = \frac{4}{3}(a_2) = -\frac{4}{3}a_0}$$

$$\sum_{n=2}^{\infty}\Big[a_n(n+r)(n+r-1) + 3a_{n+1}(n+r+1)(n+r) + 3a_{n+2}(n+r+2)(n+r+1) - a_{n+3}(n+r+3)(n+r+2)$$

$$-a_n(n+r) + 3a_{n+1}(n+r+1) - 3a_{n+2}(n+r+2) + a_{n+3}(n+r+3) + a_{n+1}\Big]x^{n+2} = 0$$

$$a_n(n-1)(n-2) + 3a_{n+1}n(n-1) + 3a_{n+2}(n+1)(n) - a_{n+3}(n+2)(n+1) - a_n(n-1)$$

$$+3a_{n+1}(n) - 3a_{n+2}(n+1) + a_{n+3}(n+2) + a_{n+1} = 0$$

$$a_n\Big[\underset{\substack{n^2-n-2-n+1\\ n^2-2n-1}}{(n+1)(n-2)-(n-1)}\Big]+a_{n+1}\Big[\underset{\substack{3n^2-3n+3n+1\\ 3n^2+1}}{3n(n-1)+3n+1}\Big]+a_{n+2}\Big[\underset{\substack{3n^2+3n-3n-3\\ 3n^2-3=3(n-1)(n}}{3(n+1)n-3(n+1)}$$ ⑨

$$+a_{n+3}\Big[\underset{\substack{-n^2-3n-2+n+2\\ -n^2-2n=-(n)(n+2)}}{-(n+2)(n+1)+n+2}\Big]=0$$

$$a_{n+3}=\frac{a_n(n^2-2n-1)}{n(n+2)}+a_{n+1}\frac{(3n^2+1)}{n(n+2)}+a_{n+2}\frac{3(n-1)(n+1)}{n(n+2)}$$

$n=2$

$$a_5=\frac{a_2(\cancel{4}-\cancel{4}-1)}{2(4)}+\frac{a_3(3\cdot4+1)}{2\cdot4}+a_4\frac{(3)(1)(3)}{2(4)}$$

$$a_5=-\frac{1}{8}(-a_0)+\frac{13}{8}\left(-\frac{5}{2}a_0-\frac{1}{2}a_1\right)+\frac{\cancel{9}^{3}}{\cancel{8}_2}\left(-\frac{4}{\cancel{3}}a_0\right)$$

$$=\frac{1}{8}a_0-\frac{65}{10}a_0-\frac{13}{16}a_1-\frac{3}{2}a_0=-\frac{63}{8}a_0-\frac{13}{16}a_1$$

$$y=\Big[a_0\left(1-x^2-\tfrac{5}{2}x^3-\tfrac{4}{3}x^4-\tfrac{63}{8}x^5-\tfrac{694}{45}x^6+\dots\right)$$
$$a_1\left(x-\tfrac{1}{2}x^3-\tfrac{13}{16}x^5-\tfrac{41}{30}x^6+\dots\right)\Big]x^{-1}$$

$n=3$

$$a_6=\frac{a_3(9-\overset{2}{6}-1)}{3(5)}+\frac{a_4(3\cdot9+1)}{3(5)}+a_5\frac{(\cancel{3})(2)(4)}{\cancel{3}(5)}$$

$$\frac{2}{15}\left(-\frac{5}{\cancel{2}}a_0-\frac{1}{\cancel{2}}a_1\right)+\frac{28}{15}\left(-\frac{4}{3}a_0\right)+\frac{8}{5}\left(-\frac{63}{\cancel{8}}a_0-\frac{13}{\cancel{16}_2}a_1\right)$$

$$-\frac{1}{3}a_0-\frac{1}{15}a_1-\frac{112}{45}a_0-\frac{63}{5}a_0-\frac{13}{10}a_1=-\frac{694}{45}a_0-\frac{41}{30}a_1$$

i. $2xy''-y'+2y=0$

$x=0$ regular singular point
or use $x=1$ as an ordinary point

$$y=\sum_{n=0}^{\infty}a_nx^{n+r}$$

$$2x\sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} - \sum_{n=1}^{\infty} a_n(n+r)x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=2}^{\infty} 2a_n(n+r)(n+r-1)x^{n+\cancel{r}-1} - \sum_{n=1}^{\infty} a_n(n+r)x^{n+\cancel{r}-1} + \sum_{n=0}^{\infty} 2a_n x^{n+\cancel{r}} = 0$$

$$\sum_{n=1}^{\infty} 2a_{n+1}(n+r+1)(n+r)x^n - \sum_{n=0}^{\infty} a_{n+1}(n+r+1)x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=1}^{\infty} 2a_{n+1}(n+r+1)(n+r)x^n - a_1(r+1)x^0 - \sum_{n=1}^{\infty} a_{n+1}(n+r+1)x^n + 2a_0x^0 + \sum_{n=1}^{\infty} 2a_nx^n = 0$$

$-a_1(r+1) + 2a_0 = 0$ letting $r=-1$ will make the entire solution 0. other values ok, so choose $r=0$

$-a_1 + 2a_0 = 0 \Rightarrow a_1 = 2a_0$

$n=0$

$$\sum_{n=1}^{\infty} \left[2a_{n+1}(n+1)(n) - a_{n+1}(n+1) + 2a_n\right] x^n = 0$$

$$a_{n+1}\left(2n(n+1) - (n+1)\right) = -2a_n$$

$$a_{n+1} = \frac{-2a_n}{(2n-1)(n+1)}$$ only one solution not two

$n=1$ $\quad a_2 = \frac{-\cancel{2}a_1}{(1)(\cancel{2})} = -1(2a_0) = -2a_0$

$n=2$ $\quad a_3 = \frac{-2a_2}{(3)(3)} = -\frac{2}{9}(-2a_0) = \frac{4}{9}a_0$

$n=3$ $\quad a_4 = \frac{-\cancel{2}a_3}{(5)(\cancel{4}_2)} = -\frac{1}{\cancel{10}_5}\left(\frac{4}{9}a_0\right) = -\frac{1}{45}a_0$

$n=4$ $\quad a_5 = \frac{-2a_4}{(7)(5)} = \frac{-2}{35}\left(-\frac{1}{45}\right)a_0 = \frac{2}{1575}a_0$

$$y = a_0\left(1 + 2x - 2x^2 + \frac{4}{9}x^3 - \frac{1}{45}x^4 + \frac{2}{1575}x^5 + \dots\right)$$

j. $2x^2y'' - xy' + (x^2+1)y = 0$ $\quad$ 0 regular singular point

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$2x^2\sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} - x\sum_{n=1}^{\infty} a_n(n+r)x^{n+r-1} + (x^2+1)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

j cont'd.

$$\sum_{n=2}^{\infty} 2a_n(n+r)(n+r-1)x^{\cancel{n+r}} - \sum_{n=1}^{\infty} a_n(n+r)x^{\cancel{n+r}} + \sum_{n=0}^{\infty} a_n x^{\cancel{n+r}+2} + \sum_{n=0}^{\infty} a_n x^{\cancel{n+r}} = 0$$ (11)

$$\sum_{n=0}^{\infty} 2a_{n+2}(n+r+2)(n+r+1)x^{n+2} - \sum_{n=-1}^{\infty} a_{n+2}(n+r+2)x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} + \sum_{n=-2}^{\infty} a_{n+2}x^{n+2}$$

$n=-1$ $\qquad$ $n=-2,-1$

$$\sum_{n=0}^{\infty} 2a_{n+2}(n+r+2)(n+r+1)x^{n+2} - a_1(r+1)x - \sum_{n=0}^{\infty} a_{n+2}(n+r+2)x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2}$$

$$+ a_0 x^0 + a_1 x + \sum_{n=0}^{\infty} a_{n+2}x^{n+2} = 0$$

$$-\cancel{a_1}(r+1) + \cancel{a_1} = 0 \Rightarrow 1 = r+1 \Rightarrow \underline{r=0} \quad \ni \quad a_0 = 0$$

$$\sum_{n=0}^{\infty}\left[2a_{n+2}(n+2)(n+1) - a_{n+2}(n+2) + a_n + a_{n+2}\right]x^{n+2} = 0$$

$$a_{n+2}\left[2(n+2)(n+1) - (n+2) + 1\right] = -a_n$$

$2(n^2+3n+2) - n - 2 + 1$

$2n^2 + 6n + 4 - n - 2 + 1$

$2n^2 + 5n + 3$

$(2n+3)(n+1)$

$$a_{n+2} = \frac{-a_n}{(2n+3)(n+1)}$$

$a_0 = 0$
all even terms 0

$n=1$ $\quad a_3 = \dfrac{-a_1}{(7)(2)} = -\dfrac{1}{14}a_1$

$n=3$ $\quad a_5 = \dfrac{-a_3}{(9)(4)} = -\dfrac{1}{36}\left(-\dfrac{1}{14}\right)a_1 = \dfrac{1}{504}a_1$

$n=5$ $\quad a_7 = \dfrac{-a_5}{(13)(6)} = -\dfrac{1}{78}\left(+\dfrac{1}{504}\right)a_1 = -\dfrac{1}{39312}a_1$

$$y = a_1\left(x - \frac{1}{14}x^3 + \frac{1}{504}x^5 - \frac{1}{39{,}312}x^7 + \cdots\right)$$

k. $xy'' - xy' + y = 0$ $\qquad$ $x=0$ regular singular point

$$x\sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^{n+\cancel{r}-2} - x\sum_{n=1}^{\infty} a_n(n+r)x^{n+\cancel{r}-1} + \sum_{n=0}^{\infty} a_n x^{n+\cancel{r}} = 0$$

$$\sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^{n-1} - \sum_{n=1}^{\infty} a_n(n+r)x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} a_{n+1}(n+r+1)(n+r)x^n - \sum_{n=1}^{\infty} a_n(n+r)x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty}\left[a_{n+1}(n+\cancel{r}+1)(n+\cancel{r}) - a_n(n+\cancel{r}) + a_n\right]x^n + a_0x^0 = 0 \qquad a_0 = 0$$

$a_n(-1)(n) + a_n(1)$ $\qquad$ no condition on r

$a_n(-n+1)$

$$a_{n+1} = \frac{a_n(n-1)}{(n+1)n}$$

k cont'd.

$n=1 \quad a_2 = \frac{a_1(0)}{(2)(1)} = 0$

$y = a_1(x)$

$n=2 \quad a_3 = \frac{a_2(1)}{(3)(2)} = 0 \cdot \frac{1}{6} = 0$

all other coefficients $= 0$

l. $x^2y'' - 2y = 0$ x=0 is a regular singular point

$$x^2 \sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} - 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^n - \sum_{n=2}^{\infty} 2a_n x^n - 2a_0x^0 - 2a_1x^1 = 0$$

$$\sum_{n=2}^{\infty} [a_n(n+r)(n+r-1) - 2a_n]x^n - 2a_0 - 2a_1 = 0$$

$a_0 = 0 \quad a_1 = 0$

$$a_n[(n+r)(n+r-1) - 2] = 0$$

$(n+r)(n+r-1) = 2$ if $r=0$

$n(n-1) = 2$

$n^2 - n - 2 = 0$ This is equation obtained from Cauchy-Euler process.

$(n-2)(n+1) = 0$

$n = 2, \; n = -1$

$$y = a_2x^2 + a_{-1}x^{-1}$$

m. $x^2y'' + 5xy' + 4y = 0$ x=0 is a regular singular point

$$x^2 \sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} + 5x\sum_{n=1}^{\infty} a_n(n+r)x^{n+r-1} + 4\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^n + \sum_{n=1}^{\infty} 5a_n(n+r)x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=2}^{\infty} [a_n(n+r)(n+r-1) + 5a_n(n+r) + 4a_n]x^n + 5a_1rx^1 + 4a_0x^0 + 4a_1x = 0$$

$a_0 = 0 \qquad 5a_1r + 4a_1 = 0 \Rightarrow a_1(5r+4) = 0 \Rightarrow r = -\frac{4}{5}$

m cont'd.

$$a_n(n-\tfrac{4}{5})(n-\tfrac{9}{5}) + 5a_n(n-\tfrac{4}{5}) + 4a_n = 0$$

$$a_n\left[(n-\tfrac{4}{5})(n-\tfrac{9}{5}) + 5(n-\tfrac{4}{5}) + 4\right] = 0 \qquad a_n = 0 \text{ or}$$

$$n^2 - \tfrac{9}{5}n - \tfrac{4}{5}n + \tfrac{36}{25} + 5n - \cancel{4} + \cancel{4} = n^2 + \tfrac{12}{5}n + \tfrac{36}{25} = 0$$

$$(n-\tfrac{6}{5})^2 = 0 \qquad n = \tfrac{6}{5}$$

not strictly geometric since n is a fractional value.

$$x^{n+r} = x^{6/5 - 4/5} = x^{2/5}$$

$$y = c_1 x^{-2} + c_2 x^{-2}\ln x$$ from Cauchy-Euler

no-whole power solutions

n. $x^3 y''' - 6y = 0$ $\qquad$ $x=0$ is a regular singular point for a cubic

$$x^3 \sum_{n=3}^{\infty} a_n (n+r)(n+r-1)(n+r-2)x^{n+r-3} - 6\sum_{n=0}^{\infty} a_n x^{n+r} =$$

$$\sum_{n=3}^{\infty} a_n (n+r)(n+r-1)(n+r-2)x^n - \sum_{n=0}^{\infty} 6a_n x^n$$

$$\sum_{n=3}^{\infty} a_n\left[(n+r)(n+r-1)(n+r-2) - 6\right]x^n - 6a_0 - 6a_1 x - 6a_2 x^2 = 0$$

$r = 0$ $\qquad$ $a_0 = 0 \quad a_1 = 0, \; a_2 = 0$

Cauchy-Euler.

$$n(n-1)(n-2) - 6 = n(n^2-3n+2) - 6 = n^3 - 3n^2 + 2n - 6 =$$

$$n^2(n-3) + 2(n-3) = 0 \Rightarrow n = 3, \quad n^2 + 2 = 0 \quad n = \pm\sqrt{2}i$$

produces non-geometric solutions

$$y = a_3 x^3 + \underbrace{c_1 \cos(\ln\sqrt{2}t) + c_2 \sin(\ln\sqrt{2}t)}_{\text{from Cauchy-Euler.}}$$

o. $(x^2+1)y'' + 2xy' = 0$ $\qquad$ no singular points

$$x^2 \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + 2x\sum_{n=1}^{\infty} a_n n x^{n-1} = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} 2a_n n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} 2a_n n x^n = 0$$

6 cont'd. ⑭

$$\sum_{n=2}^{\infty}\left[a_n n(n-1) + a_{n+2}(n+2)(n+1) + 2a_n n\right]x^n + a_2(2)(1)x^0 + a_3(3)(2)x + 2a_1(1)x^1 = 0$$

$2a_2 = 0 \Rightarrow a_2 = 0$ even series ends at a_0

$6a_3 + 2a_1 = 0 \Rightarrow a_3 = -\frac{a_1}{3}$

$$a_{n+2} = \frac{-a_n(\cancel{n} + n^2 - \cancel{n})}{(n+2)(n+1)} = \frac{-a_n(n^2)}{(n+2)(n+1)}$$

$n=3$ $\quad a_5 = \frac{-a_3(9)}{5\cdot 4} = \frac{+a_1\cdot 3}{20} = \frac{3}{20}a_1$

$n=5$ $\quad a_7 = \frac{-a_5(25)}{7\cdot 6} = -\frac{\cancel{3}\cdot \cancel{25}^{5}\, a_1}{7\cdot\cancel{6}_2\cdot\cancel{20}_4} = \frac{-5a_1}{56}$

$n=7$ $\quad a_9 = \frac{-a_7(49)}{9\cdot 8} = \frac{5a_1\cdot\cancel{49}^{7}}{\cancel{56}_8\cdot 9\cdot 8} = \frac{35a_1}{576}$

$$y = a_0 + a_1\left(x - \frac{1}{3}x^3 + \frac{3}{20}x^5 - \frac{5}{56}x^7 + \frac{35}{576}x^9 + \dots\right)$$

P. $(x^2+2)y'' + 3xy' - y = 0 \qquad x_0 = 1$

$3(x-1)+3 = 3x$

$(x-1)^2 = x^2 - 2x + 1$ $\qquad (x-1)^2 + 2x - 1 = x^2$

$2(x-1) = 2x - 2$ $\qquad (x-1)^2 + 2(x-1) + 2 - 1 = x^2$

$(x-1)^2 + 2(x-1) + 3 = x^2 + 2$

$$(x-1)^2\sum_{n=2}^{\infty}a_n n(n-1)(x-1)^{n-2} + 2(x-1)\sum_{n=2}^{\infty}a_n n(n-1)(x-1)^{n-2} + 3\sum_{n=2}^{\infty}a_n n(n-1)(x-1)^{n-2}$$

$$+ 3(x-1)\sum_{n=1}^{\infty}a_n n(x-1)^{n-1} + 3\sum_{n=1}^{\infty}a_n n(x-1)^{n-1} - \sum_{n=0}^{\infty}a_n(x-1)^n = 0$$

$$\sum_{n=2}^{\infty}a_n n(n-1)(x-1)^n + \sum_{n=2}^{\infty}2a_n n(n-1)(x-1)^{n-1} + \sum_{n=2}^{\infty}3a_n n(n-1)(x-1)^{n-2} + \sum_{n=1}^{\infty}3a_n n(x-1)^n$$

$$+ \sum_{n=1}^{\infty}3a_n n(x-1)^{n-1} - \sum_{n=0}^{\infty}a_n(x-1)^n = 0$$

p cont'd.

$$\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^n + \sum_{n=1}^{\infty} 2a_{n+1}(n+1)(n)(x-1)^n + \sum_{n=0}^{\infty} 3\, a_{n+2}(n+2)(n+1)(x-1)^n +$$

$$\sum_{n=1}^{\infty} 3a_n n\,(x-1)^n + \sum_{n=0}^{\infty} 3a_{n+1}(n+1)(x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=2}^{\infty} \left[a_n n(n-1) + 2a_{n+1}(n+1)(n) + 3a_{n+2}(n+2)(n+1) + 3a_n n + 3a_{n+1}(n+1) - a_n\right]$$

$$(x-1)^n + 2a_2(2)(1)x^1 + 3a_2(2)(1)x^0 + 3a_3(3)(2)x^1 + 3a_1(1)x^1 +$$

$$3a_1(1)x^0 + 3a_2(2)x^1 - a_0x^0 - a_1x^1 = 0$$

$$a_n[\underbrace{n(n-1)+3n-1}_{\substack{n^2-n+3n-1 \\ n^2+2n-1}}] + a_{n+1}[\underbrace{2(n+1)(n)+3(n+1)}_{\substack{2n^2+2n+3n+3 \\ 2n^2+5n+3 \\ (2n+3)(n+1)}}] + a_{n+2}[3(n+2)(n+1)] = 0$$

$$a_{n+2} = -\frac{a_n(n^2+2n-1)}{3(n+2)(n+1)} - \frac{a_{n+1}(2n+3)\cancel{(n+1)}}{3(n+2)\cancel{(n+1)}}$$

$$6a_2 + 3a_1 - a_0 = 0 \Rightarrow 6a_2 = a_0 - 3a_1 \Rightarrow \boxed{a_2 = \tfrac{1}{6}a_0 - \tfrac{1}{2}a_1}$$

$$4a_2 + 18a_3 + 3a_1 + 6a_2 - a_1 = 0 \Rightarrow 10a_2 + 18a_3 + 2a_1 = 0$$

$$\boxed{a_3 = -\tfrac{5}{9}a_2 - \tfrac{1}{9}a_1} = -\tfrac{5}{9}\left(\tfrac{1}{6}a_0 - \tfrac{1}{2}a_1\right) - \tfrac{1}{9}a_1 = -\tfrac{5}{54}a_0 - \tfrac{5}{18}a_1 - \tfrac{1}{9}a_1 =$$

$$-\tfrac{5}{54}a_0 - \tfrac{1}{6}a_1$$

$n=2$ $$a_4 = \frac{-a_2(\overset{7}{4+4-1})}{3(4)(3)} - \frac{a_3(7)}{3(4)} = -\frac{7}{36}\left(\tfrac{1}{6}a_0 - \tfrac{1}{2}a_1\right) - \frac{7}{12}\left(-\tfrac{5}{54}a_0 - \tfrac{1}{6}a_1\right)$$

$$= \frac{7}{216}a_0 - \cancel{\frac{7}{72}a_1} + \frac{35}{648}a_0 + \cancel{\frac{7}{72}a_1} = \frac{7}{81}a_0$$

$n=3$ $$a_5 = -\frac{a_3(\overset{14\,7}{9+6-1})}{3(5)(4)^2} - \frac{a_4(\overset{3}{\cancel{9}})}{\cancel{3}\cdot 5} = \frac{7}{30}\left(-\tfrac{5}{54}a_0 - \tfrac{1}{6}a_1\right) - \frac{3}{5}\left(\frac{7}{\underset{27}{\cancel{81}}}\right)a_0 =$$

$$-\frac{7}{324}a_0 - \frac{7}{180}a_1 - \frac{7}{135}a_0 = -\frac{7}{180}a_1 - \frac{119}{1620}a_0$$

$$y = a_0\left(1 + \tfrac{1}{6}x^2 - \tfrac{5}{54}x^3 + \tfrac{7}{81}x^4 - \tfrac{119}{1620}x^5 + \dots\right)$$

$$+ a_1\left(x - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 - \tfrac{7}{180}x^5 + \dots\right)$$