Name KEY

Math 2568, Exam #1 – Part 1, Spring 2013

Instructions: On this portion of the exam, you may **NOT** use a calculator. Show all work. Answers must be supported by work to receive full credit.

1. Given the system of equations $\begin{cases} x_1 \quad\quad - 5x_3 = 1 \\ 2x_1 + 4x_2 \quad\quad = 10 \\ \quad - 3x_2 - 6x_3 = -3 \end{cases}$, write the *system* as:

   a. An augmented matrix (2 points)

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 2 & 4 & 0 & 10 \\ 6 & -3 & -6 & -3 \end{array}\right]$$

   b. A vector equation (2 points)

$$x_1\begin{bmatrix}1\\2\\0\end{bmatrix} + x_2\begin{bmatrix}0\\4\\-3\end{bmatrix} + x_3\begin{bmatrix}-5\\0\\-6\end{bmatrix} = \begin{bmatrix}1\\10\\-3\end{bmatrix}$$

   c. A matrix equation. (2 points)

$$\begin{bmatrix}1 & 0 & -5\\2 & 4 & 0\\0 & -3 & -6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1\\10\\-3\end{bmatrix}$$

   d. Solve the system using the augmented matrix and row operations. State whether the solution of the system ~~is~~ consistent or inconsistent. If the system is consistent, state whether it is independ~~ent or dep~~endent. Write an independent solution in vector form; write a depen~~dent solution~~ in parametric form. (5 points)

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 2 & 4 & 0 & 10 \\ 0 & -3 & 6 & -3 \end{array}\right] \quad -2R_1+R_2 \to R_2 \quad \begin{array}{cccc} -2 & 0 & 10 & -2 \\ 2 & 4 & 0 & 10 \\ \hline 0 & 4 & 10 & 8 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 4 & 10 & 8 \\ 0 & -3 & 6 & -3 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\boxed{\vec{x} = \begin{bmatrix}11\\-3\\2\end{bmatrix}}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & 4 & 10 & 8 \end{array}\right] \quad R_2(-1/3) \to R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 4 & 10 & 8 \end{array}\right] \quad -4R_2+R_3 \to R_3 \quad \begin{array}{cccc} 0 & -4 & -8 & -4 \\ 0 & 4 & 10 & 8 \\ \hline 0 & 0 & 2 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & 4 \end{array}\right] \quad \tfrac{1}{2}R_3 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{array}{c} 2R_3+R_2 \to R_2 \\ 5R_3+R_1 \to R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

2. Given $A=\begin{bmatrix}3 & 7\\ -2 & 1\end{bmatrix}$, find A^{-1}. (5 points)

$A^{-1}=\frac{1}{3+14}\begin{bmatrix}1 & -7\\ 2 & 3\end{bmatrix}=\begin{bmatrix}1/17 & -7/17\\ 2/17 & 3/17\end{bmatrix}$

$\left[\begin{array}{cc|cc}3 & 7 & 1 & 0\\ -2 & 1 & 0 & 1\end{array}\right]$ $\frac{1}{3}R_1 \to R_1$ $\left[\begin{array}{cc|cc}1 & 7/3 & 1/3 & 0\\ -2 & 1 & 0 & 1\end{array}\right]$ $2R_1+R_2 \to R_2$ $\left[\begin{array}{cc|cc}1 & 7/3 & 1/3 & 0\\ 0 & 17/3 & 2/3 & 1\end{array}\right]$

$\frac{3}{17}R_2 \to R_2$ $\left[\begin{array}{cc|cc}1 & 7/3 & 1/3 & 0\\ 0 & 1 & 2/17 & 3/17\end{array}\right]$ $-\frac{7}{3}R_2+R_1 \to R_1$ $\left[\begin{array}{cc|cc}1 & 0 & 1/17 & -7/17\\ 0 & 1 & 2/17 & 3/17\end{array}\right]$.

3. Given $A=\begin{bmatrix}3 & 0\\ -1 & 5\end{bmatrix}, B=\begin{bmatrix}0 & 5 & -1\\ 4 & -2 & 0\end{bmatrix}, C=\begin{bmatrix}9\\ -8\\ 4\end{bmatrix}$, compute the following, if possible. If the combination is not possible, briefly explain why. (3 points each)

a) AB

$\underset{2\times2}{\begin{bmatrix}3 & 0\\ -1 & 5\end{bmatrix}}\underset{2\times3}{\begin{bmatrix}0 & 5 & -1\\ 4 & -2 & 0\end{bmatrix}} = 2\times3$

$=\begin{bmatrix}0 & 15 & -3\\ 20 & -15 & 1\end{bmatrix}$

b) CB $(3\times1)(2\times3)$ not defined

dimensions don't match

c) B^T $=\begin{bmatrix}0 & 4\\ 5 & -2\\ -1 & 0\end{bmatrix}$

f) $3I_2$ + A

$3I_2 = 3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$

$\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}+\begin{bmatrix}3 & 0\\ -1 & 5\end{bmatrix}=\begin{bmatrix}6 & 0\\ -1 & 8\end{bmatrix}$

4. Use matrix multiplication to determine if $\vec{x} = \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}$ is a solution to the system

$$\begin{cases} x_1 \quad\quad -3x_3 = 9 \\ 2x_1 - 2x_2 - 7x_3 = 10 \\ \quad -x_2 - 5x_3 = 6 \end{cases}$$. (5 points)

$$\begin{bmatrix} 1 & 0 & -3 \\ 2 & -2 & -7 \\ 0 & -1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 3+0+6 \\ 6-10+14 \\ 0-5+10 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \\ 5 \end{bmatrix}$$

No, it isn't a solution.

5. Graph the vectors $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $\vec{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and label which vector is which on the graph. On the same graph also plot the following and label each part clearly: (10 points)

a. $\vec{u} + \vec{v}$

b. $3\vec{u} - 2\vec{v}$ $\quad \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ -6 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$

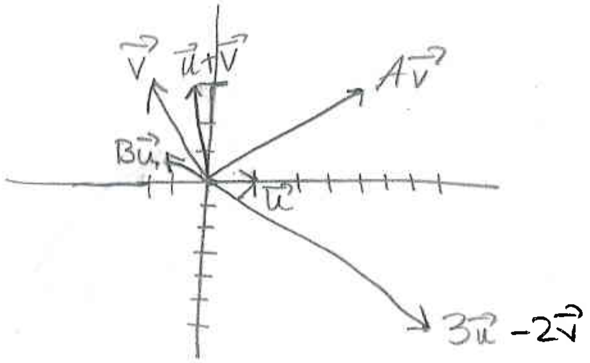
c. For $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, plot $A\vec{v}$

d. Rotate $\vec{u}$ through a counterclockwise angle of $\frac{3\pi}{4}$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2+6 \\ 0+3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta = 3\pi/4$$

$$\Rightarrow \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$



6. Determine if each statement is True or False. (2 points each)

a. T F Every linear transformation on a finite vector space is a matrix transformation and every matrix transformation is a linear transformation.

b. T F If A is a $m \times n$ matrix that has n pivot columns, then the equation $A\mathbf{x} = \mathbf{b}$ is unique for all $\mathbf{b}$ in $\Re^m$. → linearly independent

true for all b in the span

c. T F If A is a 3x3 matrix, then the transformation $\vec{x} \mapsto A\vec{x}$ must be one-to-one and onto.

d. T F Matrix multiplication is associative.

e. T F The result of multiplication between a 2x3 matrix and a 3x2 matrix results in a 3x3 matrix.

f. T F If a system of equations has a free variable then it has a unique solution.

g. T F If A is a 2x2 matrix for a projection transformation, it is not invertible.

h. T F The equation $\vec{x} = \vec{p} + t\vec{v}$ describes a line through $\vec{p}$ parallel to $\vec{v}$.

i. T F $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 11 \end{bmatrix}$ form a linearly independent set.

j. T F The mapping defined by the differential operator $\frac{d}{dx}$ is a linear transformation.

k. T F The pivot positions in a matrix depend on whether row interchanges take place.

l. T F The linear transformation given by A= $\begin{bmatrix} 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$ is onto.

Name KEY

Math 2568, Exam #1 – Part 2, Spring 2013

Instructions: On this portion of the exam, you *may* use a calculator to perform elementary matrix operations. Support your answers with work (reproduce the reduced matrices from your calculator) or other justification for full credit.

1. Find the general solution to the system $\begin{cases} x_1 - 2x_2 + 4x_3 + 5x_4 = 2 \\ -x_1 + x_2 - 3x_3 + x_4 = 7 \end{cases}$. State whether the solution of the system is consistent or inconsistent. If the system is consistent, state whether it is independent or dependent. Write an independent solution in vector form; write a dependent solution in parametric form. Circle the pivots of the reduced matrix. (6 points)

$$\text{rref} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & -7 & -16 \\ 0 & 1 & -1 & -6 & -9 \end{array}\right]$$

free free

$x_1 = -16 - 2t + 7s$

$x_2 = -9 + t + 6s$

$x_3 = t$

$x_4 = s$

$$\vec{x} = \begin{bmatrix} -16 \\ -9 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

Consistent
dependent

2. Determine if $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -6 \end{bmatrix}$ is in the span of the columns of $A = \begin{bmatrix} -1 & 2 & 1 & 5 \\ 3 & 0 & 2 & 2 \\ -4 & 2 & 5 & 9 \\ 1 & 3 & 0 & 6 \end{bmatrix}$. If it is, write **b** as a linear combination of the columns of *A*; if not, explain why it is not, and give an example of a vector that *is* in the span. (6 points)

$$\text{rref} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

inconsistent

b is not in the span of A

$$-1\begin{bmatrix} -1 \\ 3 \\ -4 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 0 \\ 2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -9 \\ -7 \\ 5 \end{bmatrix}$$

3. Let A = $\begin{bmatrix} 1 & 3 & 0 & 9 & 8 \\ -1 & -4 & 2 & -7 & 0 \\ 0 & 6 & 1 & -1 & 0 \\ 2 & 2 & 3 & 0 & 1 \\ 7 & 0 & -5 & -3 & -1 \\ 1 & 0 & 11 & 2 & 4 \end{bmatrix}$ rref $\rightarrow$ $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ 5 pivots

a. Determine if the columns of A form a linearly independent or dependent set and justify your answer. (4 points)

yes, the column vectors are linearly independent Since there are 5 vectors and 5 pivots

b. Determine if the columns of A span $\mathbb{R}^6$. Justify your answer. (4 points)

no, it spans a 5-dimensional subspace of $\mathbb{R}^6$ Since there are 5 pivots and not 6.

c. Use the information obtained in parts a and b to determine if the linear transformation $T: \vec{x} \in R^5 \mapsto A\vec{x} \in R^6$ is one-to-one or onto. Justify your answer. (4 points)

the transformation is one-to-one since the column vectors are independent, but it is not onto since it doesn't span $\mathbb{R}^6$.

4. **Use an inverse matrix** to solve $\begin{cases} x_1 \quad\quad - 2x_3 = 1 \\ -3x_1 + x_2 + 4x_3 = -5 \\ 2x_1 - 3x_2 + 4x_3 = 8 \end{cases}$. Give the inverse matrix used. (6 points)

$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

$A\vec{x} = \vec{b} \Rightarrow$
$\vec{x} = A^{-1}\vec{b}$

$A^{-1}\vec{b} = \boxed{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \vec{x}}$

$A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$

5. Not all linear transformations can be written as matrices, such as the derivative operator, because they operate on an infinite dimensional vector space (the set of all possible functions); however, if we limit such operators to a finite dimensional space, we can write the linear operator as a matrix. Consider the space P_3 defined as the set of all polynomials of degree 3 or less. These polynomials of the form $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ can be written as a 4-dimensional vector, since all their components can be determined by a set of 4 constants.

   a. Write the general polynomial p(t) above as a vector in R^4. (3 points)

$$\vec{p} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

   b. Take the derivative of p(t) and write the resulting vector (now in $P_2 \sim R^3$). (3 points)

$$p'(t) = a_1 + 2a_2 t + 3a_3 t$$

$$\vec{p}' = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \end{bmatrix}$$

   c. Create a matrix linear transformation capable of transforming the vector in part a to the vector in part b, i.e. find A such that $\vec{p} \mapsto A\vec{p} = \vec{p}'$. (5 points)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \end{bmatrix}$$

$3\times4 \quad 4\times1 \quad = \quad 3\times1$

$T: \vec{p} \in \mathbb{R}^4 \to p' \in \mathbb{R}^3$

6. The invertible matrix theorem states that several statements are equivalent to matrix A being invertible. Name 4 of these equivalent statements (so far there are 11 to choose from). (8 points)

Answers will vary. See theorem in the text for complete list:

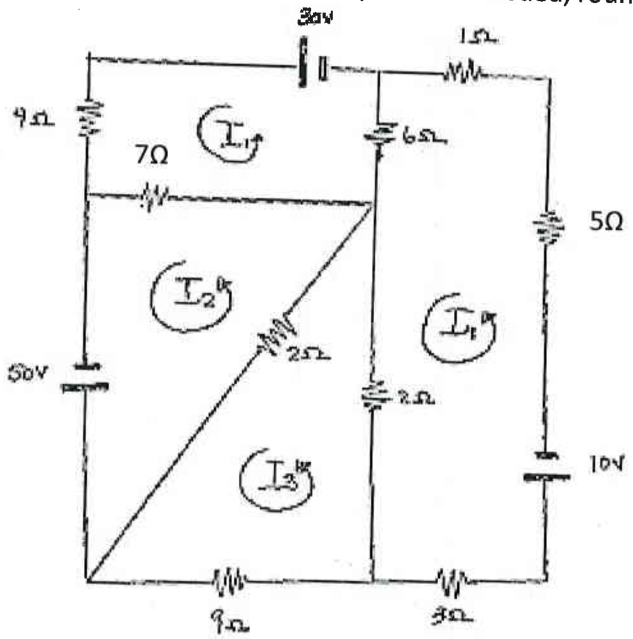
example: ① $A\vec{x} = \vec{0}$ only trivial solution

② A^T is invertible

③ A spans R^n

④ column vectors of A are linearly independent.

7. Using the current diagram below, create the system of equations needed to solve for all 4 loop currents. Solve the system. If needed, round the current values to two decimal places. (8 points)



$$17I_1 - 7I_2 \qquad -6I_4 = 30$$
$$-7I_1 + 9I_2 - 2I_3 \qquad = 50$$
$$-2I_2 + 19I_3 - 8I_4 = 0$$
$$-6I_1 \qquad -8I_3 + 23I_4 = -10$$

$$\left[\begin{array}{cccc|c} 17 & -7 & 0 & -6 & 30 \\ -7 & 9 & -2 & 0 & 50 \\ 0 & -2 & 19 & -8 & 0 \\ -6 & 0 & -8 & 23 & -10 \end{array}\right]$$

$$\vec{I} = \begin{bmatrix} 7.44 \\ 11.83 \\ 2.20 \\ 2.27 \end{bmatrix}$$

8. Prove that the transformation T defined by the $T: \vec{x} \mapsto A\vec{x}$ for the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ is linear using the definition. (9 points)

① $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix} = \begin{bmatrix} u_2+v_2 \\ u_1+v_1+u_3+v_3 \\ u_2+v_2+2u_3+2v_3 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 \\ u_1+u_3 \\ u_2+2u_3 \end{bmatrix} + \begin{bmatrix} v_2 \\ v_1+v_3 \\ v_2+2v_3 \end{bmatrix} = \begin{bmatrix} u_2+v_2 \\ u_1+u_3+v_1+v_3 \\ u_2+v_2+2u_3+2v_3 \end{bmatrix}$ ✓

② $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} = \begin{bmatrix} cu_2 \\ cu_1+cu_3 \\ cu_2+2cu_3 \end{bmatrix}$ $\quad c\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = c\begin{bmatrix} u_2 \\ u_1+u_3 \\ u_2+2u_3 \end{bmatrix} = \begin{bmatrix} cu_2 \\ cu_1+cu_3 \\ cu_2+2cu_3 \end{bmatrix}$ ✓

③ $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

9. Answer the following questions as fully as possible, and justify your answer. (3 points each)

a. Explain why an nxn matrix can be both one-to-one and onto, but an mxn matrix where m≠n cannot be.

an nxn matrix can have n pivots & be linearly independent & therefore one-to-one, and also n pivots in rows and therefore span $\mathbb{R}^n$ and be onto. and mxn matrix (w/ m≠n) has one dimensional smaller than the other & so will come up short on one of these.

b. Use general matrix properties to show that $(ABC)^T = C^T B^T A^T$.

$$[(AB)C]^T = C^T(AB)^T = C^T(B^T A^T) = C^T B^T A^T$$

c. If A is a 5x3 matrix with three pivot positions, does the equation $A\vec{x} = \vec{0}$ have a solution? If so, is it trivial or non-trivial

yes. homogeneous equations always have a solution, at least $\vec{x} = \vec{0}$. a matrix w/3 columns & 3 pivots is linearly independent and so the only solution is the trivial one.

d. Determine if the matrix $\begin{bmatrix} -5 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix}$ is invertible. Explain why or why not.

It is not because of the row of zeros. This transformation is not one-to-one.