Name KEY

Math 2568, Exam #3 – Part 1, Spring 2013

Instructions: On this portion of the exam, you may **NOT** use a calculator. Show all work. Answers must be supported by work to receive full credit.

1. Find the eigenvalues and eigenvectors of the matrices below. Be sure to clearly indicate the characteristic equation, and which eigenvalues and eigenvectors go together. (12 points)

   a. $A = \begin{bmatrix} -2 & 5 \\ 7 & 0 \end{bmatrix}$ $(-2-\lambda)(-\lambda)-35 = \lambda^2+2\lambda-35=0$ char. eq.

$(\lambda+7)(\lambda-5)=0$

$\lambda_1=-7, \quad \lambda_2=+5$

$\begin{bmatrix} 5 & 5 \\ 7 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$x_1+x_2=0 \Rightarrow x_1=-x_2$
$x_2=x_2$

$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -7 & 5 \\ 7 & -5 \end{bmatrix} \therefore 7x_1-5x_2=0$

$x_1=\frac{5}{7}x_2$
$x_2=x_2$

$\vec{v}_2 = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

   b. $B = \begin{bmatrix} 9 & 4 \\ -26 & -11 \end{bmatrix}$ $(9-\lambda)(-11-\lambda)+104 =$

$-99-9\lambda+11\lambda+\lambda^2+104 =$

$\lambda^2+2\lambda+5=0$ char. eq.

$\lambda = \frac{-2 \pm \sqrt{4-4(5)}}{2} = \frac{-2\pm\sqrt{-16}}{2} = \frac{-2\pm 4i}{2}$

$= -1 \pm 2i \qquad \lambda_1=-1+2i \qquad \lambda_2=-1-2i$

$\begin{bmatrix} 9-(-1+2i) & 4 \\ -26 & -11-(-1+2i) \end{bmatrix} = \begin{bmatrix} 10-2i & 4 \\ -26 & -10-2i \end{bmatrix}$ $\quad -26x_1+(-10-2i)x_2=0$

$x_1 = \frac{10+2i}{-26}x_2 \qquad \vec{v}_1 = \begin{bmatrix} -\frac{5}{13}-\frac{1}{13}i \\ 1 \end{bmatrix}$
$x_2=x_2$

$\vec{v}_1 = \begin{bmatrix} -5 \\ 13 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i \qquad \vec{v}_2 = \begin{bmatrix} -5 \\ 13 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$

2. For each of the matrices above, find a similarity transformation matrix P such that a matrix with real eigenvalues can be diagonalized (i.e. $A = PDP^{-1}$, where D is diagonal), or a matrix P such that a matrix with complex eigenvalues can be written as a scaled rotation matrix (i.e. $A = PCP^{-1}, where\ C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$). Be sure to clearly indicate P as well as D or C. (12 points)

a.

$P = \begin{bmatrix} -1 & 5 \\ 1 & 7 \end{bmatrix}$ $D = \begin{bmatrix} -7 & 0 \\ 0 & 5 \end{bmatrix}$

or $P = \begin{bmatrix} 5 & -1 \\ 7 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 5 & 0 \\ 0 & -7 \end{bmatrix}$

b. $P = \begin{bmatrix} -5 & -1 \\ 13 & 0 \end{bmatrix}$ $C = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$

or

$P = \begin{bmatrix} -5 & 1 \\ 13 & 0 \end{bmatrix}$ $C = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}$

$m \times n$

3. Suppose matrix A is a 7x9 matrix with 5 pivot columns. Determine the following. (12 points)

dim Col A = 5　　　　dim Nul A = 4

dim Row A = 5　　　　If Col A is a subspace of $\mathbb{R}^m$, then m = 7

Rank A = 5　　　　If Nul A is a subspace of $\mathbb{R}^n$, then n = 9

4. Given the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ -1 \\ -5 \\ 2 \end{bmatrix}$ find the following.

a. $\mathbf{u} \cdot \mathbf{v}$ (3 points)

$$-3+3-10+10 = 0$$

b. $\|\vec{u}\|$. (3 points)

$$\sqrt{1^2+3^2+2^2+5^2} = \sqrt{1+9+4+25} = \sqrt{39}$$

c. A unit vector in the direction of **v**. (3 points)

$$\|v\| = \sqrt{9+1+25+4} = \sqrt{39}$$

$$\frac{\vec{V}}{\|v\|} = \begin{bmatrix} -3/\sqrt{39} \\ -1/\sqrt{39} \\ -5/\sqrt{39} \\ 2/\sqrt{39} \end{bmatrix}$$

d. Are **u** and **v** orthogonal? Why or why not? (3 points)

Yes, dot product is 0

5. Determine if each statement is True or False. (2 points each)

a. T F Two eigenvectors corresponding to the same eigenvalue are always linearly dependent.

not if eigenvalue is repeated

b. T F An nxn matrix can have more than n eigenvalues. n is max

c. T F If A and B are row equivalent, then their column spaces are the same. row space yes.

d. T F The rank of a matrix is defined by the dimension of the null space.

e. T F A linearly independent set in a subspace H is a basis for H. must also span

f. T F The equilibrium vector for a stochastic matrix is always unique. false can have 2 if eigenv $\lambda = 1$ is repeated

g. T F A matrix is not invertible if and only if 0 is an eigenvalue of A.

h. T F The eigenvalues of a matrix are on its main diagonal. only if triangular

i. T F The columns of an invertible *nxn* matrix form a basis for R^n.

j. T F If B is an echelon form of a matrix A, then the pivot ~~columns~~ rows of B for a basis for Row A.

k. T F The nullspace of A is the same as the column space of A^T. $\text{Col } A^T = \text{row } A$

l. T F The columns of the change-of-coordinate matrix $\underset{C \leftarrow B}{P}$ are B-coordinate vectors of the vectors in C.

m. T F The elementary row operations of A do not change its eigenvalues.

n. T F If A is diagonalizable, then A is invertible. could have $\lambda = 0$

o. T F The complex eigenvalues of a discrete dynamical system all attract to the origin.

$\|\lambda\|$ could be > 1

Name ______________________________

Math 2568, Exam #3 – Part 2, Spring 2013

Instructions: On this portion of the exam, you *may* use a calculator to perform elementary matrix operations. Support your answers with work (reproduce the reduced matrices from your calculator) or other justification for full credit.

1. In a certain region, about 15% of a city's population moves to the surrounding suburbs each year, and about 6% of the suburban population moves into the city. In 2012, 45% of the population lived in the city and 55% lived in the suburbs.
   a. Give the stochastic matrix that describes how the population tends to change each year. Give the initial state vector. (6 points)

$$P = \begin{bmatrix} .85 & .06 \\ .15 & .94 \end{bmatrix} \quad X_0 = \begin{bmatrix} .45 \\ .55 \end{bmatrix}$$

   b. What percentage of the population will live in the city in 2013? (4 points)

$$X_1 = \begin{bmatrix} .4155 \\ .5845 \end{bmatrix} \quad 41.55\%$$

   c. Eventually, what percentage of people will live in the suburbs? Give the equilibrium vector and be sure to clearly interpret the vector in light of the context. (4 points)

$$P^{360} = \begin{bmatrix} 2/7 & 2/5 \\ 5/7 & 5/7 \end{bmatrix} \Rightarrow q = \begin{bmatrix} 2/7 \\ 5/7 \end{bmatrix}$$

$$5/7 = 71.43\%$$

2. Given the bases B={b_1, b_2, b_3} and C={c_1, c_2, c_3} below, find the change of basis matrices $\underset{C\leftarrow B}{P}$ and $\underset{B\leftarrow C}{P}$. If the B-coordinate vector for $\vec{x}$ is as shown, find the C-coordinate vector for $\vec{x}$. (10 points)

$$\vec{b_1} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \vec{b_2} = \begin{bmatrix} 1 \\ 6 \\ -5 \end{bmatrix}, \vec{b_3} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \vec{c_1} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}, \vec{c_2} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \vec{c_3} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \left[\vec{x}\right]_B = \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix}$$

$$P_C = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -3 & -1 \\ 7 & 5 & 2 \end{bmatrix} \quad P_B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 6 & -1 \\ -2 & -5 & 4 \end{bmatrix}$$

$$\underset{C\leftarrow B}{P} = P_C^{-1} P_B = \begin{bmatrix} 0 & 4/3 & 2/3 \\ 0 & -1/3 & 4/3 \\ 1 & -19/3 & -11/3 \end{bmatrix}$$

$$\underset{B\leftarrow C}{P} = P_B^{-1} P_C = \begin{bmatrix} -29/6 & -1/3 & -1 \\ 2/3 & -1/3 & 0 \\ 1/6 & 2/3 & 0 \end{bmatrix}$$

$$[x]_C = P_C^{-1} P_B [x]_B$$

$$[x]_C = \begin{bmatrix} 16/3 \\ 32/3 \\ -91/3 \end{bmatrix}$$

3. For the stochastic matrix $\begin{bmatrix} .3 & .35 & .3 \\ .5 & .4 & .25 \\ .2 & .25 & .45 \end{bmatrix}$, find the steady state vector for the system. (7 points)

$$P^{90} = \begin{bmatrix} 107/335 & 107/335 & 107/335 \\ 26/67 & 26/67 & 26/67 \\ 98/335 & 98/335 & 98/335 \end{bmatrix}$$

$$q = \begin{bmatrix} 107/335 \\ 26/67 \\ 98/335 \end{bmatrix}$$

4. List at least 8 properties of Invertible Matrices from the Invertible Matrix Theorem. (8 points)

See textbook for a complete list however examples include

1) A is invertible
2) A^T is invertible
3) A does not have an eigenvalue $\lambda = 0$
4) $\det A \neq 0$
5) n pivots
6) row-reduces to identity
7) $\text{Nul } A = \{0\}$
8) $\dim \text{Nul } A = 0$
9) $\text{rank } A = n$

etc.

5. Consider the discrete dynamical system given by the matrix $A = \begin{bmatrix} .4 & .5 \\ -.85 & 1.2 \end{bmatrix}$.

a. Determine the behaviour of the origin for this system: is it a repeller, an attractor or a saddle point? (10 points)

$$\begin{bmatrix} .4-\lambda & .5 \\ -.85 & 1.2-\lambda \end{bmatrix} = (.4-\lambda)(1.2-\lambda) + .425 = \lambda^2 - 1.6\lambda + .905$$

$$\lambda = \frac{1.6 \pm \sqrt{-1.06}}{2} \qquad \lambda \approx .8 \pm .51i \qquad \| .8 \pm .51 i \| \approx .95 < 1$$

attractor

b. Given the initial condition of the population as $x_0 = \begin{bmatrix} 30 \\ 13 \end{bmatrix}$, find 10 points of the trajectory for the system. Are the populations still alive when the 10th sample is taken? (10 points)

$$x_0 = \begin{bmatrix} 30 \\ 13 \end{bmatrix}, x_1 = \begin{bmatrix} 18.5 \\ -9.9 \end{bmatrix}, x_2 = \begin{bmatrix} 2.45 \\ -27.6 \end{bmatrix}, x_3 = \begin{bmatrix} -12.8 \\ -35.2 \end{bmatrix}, x_4 = \begin{bmatrix} -22.7 \\ -31.35 \end{bmatrix}, x_5 = \begin{bmatrix} -24.8 \\ -18.3 \end{bmatrix}$$

$$x_6 = \begin{bmatrix} -19 \\ -.9 \end{bmatrix}, x_7 = \begin{bmatrix} -8. \\ 15.1 \end{bmatrix}, x_8 = \begin{bmatrix} 4.3 \\ 25 \end{bmatrix}, x_9 = \begin{bmatrix} 14.2 \\ 26.3 \end{bmatrix}, x_{10} = \begin{bmatrix} 18.8 \\ 19.49 \end{bmatrix}$$

x_{10} has both values positive, hower They were negative for a time So if the population is not modeled by some Constant $> 35 + x_0$ then the real world population would have collapsed

6. Define $T: R^2 \to R^2$ by $T(\vec{x}) = A\vec{x}$. Find a basis $\mathscr{B}$ for R^2 with the property that $[T]_B$ is diagonal, given $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$. (8 points)

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{vmatrix} = (1-\lambda)(-4-\lambda) - 6 = \lambda^2 + 3\lambda - 10 = 0$$

$$(\lambda + 5)(\lambda - 2) = 0$$

$$\lambda = -5, \lambda = 2$$

$$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \quad 3x_1 + x_2 = 0$$

$$x_1 = -\frac{1}{3}x_2 \qquad \vec{v}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$x_2 = x_2$$

$$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \quad -x_1 + 2x_2 = 0$$

$$x_1 = 2x_2 \qquad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$x_2 = x_2$$

$$[T]_B = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$$

and basis required is $B = \left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$