Name KEY

Math 2568, Final Exam – Part 1, Spring 2013

Instructions: On this portion of the exam, you may **NOT** use a calculator. Show all work. Answers must be supported by work to receive full credit.

1. Compute A - 2B given $A = \begin{bmatrix} 2 & 1 & 5 \\ -1 & 0 & 6 \\ 2 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 7 & 1 \\ 0 & 2 & 8 \\ -6 & 1 & 0 \end{bmatrix}$ (8 points)

$$\begin{bmatrix} 2-2(-1) & 1-2(7) & 5-2(1) \\ -1-2(0) & 0-2(2) & 6-2(8) \\ 2-2(-6) & 1-2(1) & -1-2(0) \end{bmatrix} = \begin{bmatrix} 4 & -13 & 3 \\ -1 & -4 & -10 \\ 14 & -1 & -1 \end{bmatrix}$$

2. Find the determinant by any means. $\begin{vmatrix} 1 & -1 & 5 \\ 3 & 1 & -3 \\ 7 & -2 & 0 \end{vmatrix}$ (15 points)

$$5\begin{vmatrix} 3 & 1 \\ 7 & -2 \end{vmatrix} - (-3)\begin{vmatrix} 1 & -1 \\ 7 & -2 \end{vmatrix} + \cancel{0\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}}$$

$$5(-6-7) + 3(-2+7) = 5(-13) + 3(+5) = -65 + 15 =$$

$$-50$$

3. Find the distance between the vectors $\begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}$. (6 points)

$$\sqrt{(5-1)^2 + (0-(-2))^2 + (7-(-4))^2} = \sqrt{4^2 + 2^2 + 11^2} =$$

$$\sqrt{16 + 4 + 121} = \sqrt{141}$$

4. Given the system of equations $\begin{cases} 7x_1 - x_2 - 3x_3 = 16 \\ -x_1 \quad\quad - 5x_3 = -9 \\ \quad -2x_2 + 4x_3 = 8 \end{cases}$, write the *system* as:

a. An augmented matrix (5 points)

$$\left[\begin{array}{ccc|c} 7 & -1 & -3 & 16 \\ -1 & 0 & -5 & -9 \\ 0 & -2 & 4 & 8 \end{array}\right]$$

b. A vector equation (5 points)

$$x_1\begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 16 \\ -9 \\ 8 \end{bmatrix}$$

c. A matrix equation. (5 points)

$$\begin{bmatrix} 7 & -1 & -3 \\ -1 & 0 & -5 \\ 0 & -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16 \\ -9 \\ 8 \end{bmatrix}$$

d. Solve the system using the augmented matrix and row operations. State whether the solution of the system is consistent or inconsistent. If the system is consistent, state whether it is independent or dependent. Write an independent solution in vector form; write a dependent solution in parametric form. (15 points)

$$R_1 \leftrightarrow R_2 \quad \left[\begin{array}{ccc|c} -1 & 0 & -5 & -9 \\ 7 & -1 & -3 & 16 \\ 0 & -2 & 4 & 8 \end{array}\right] \quad 7R_1 + R_2 \to R_2 \quad \left[\begin{array}{ccc|c} -1 & 0 & -5 & -9 \\ 0 & -1 & -38 & -47 \\ 0 & -2 & 4 & 8 \end{array}\right]$$

$$\begin{array}{l} -R_1 \to R_1 \\ -R_2 \to R_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 5 & 9 \\ 0 & 1 & 38 & 47 \\ 0 & -2 & 4 & 8 \end{array}\right] \quad 2R_2 + R_3 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 0 & 5 & 9 \\ 0 & 1 & 38 & 47 \\ 0 & 0 & \frac{80}{80} & \frac{102}{80} \end{array}\right] = 51/40$$

$x_3 = 51/40$

$x_2 = -29/20$

$x_1 = 21/8$

$$\boxed{\vec{x} = \begin{bmatrix} 21/8 \\ -29/20 \\ 51/40 \end{bmatrix}}$$

$47 - 38(51/40) = x_2$

$9 - 5(51/40) = x_1$

5. Find the inverse of $\begin{bmatrix} 5 & 8 \\ 4 & 7 \end{bmatrix}$ (8 points)

$$\begin{bmatrix} 7/3 & -8/3 \\ -4/3 & 5/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 7 & -8 \\ -4 & 5 \end{bmatrix}$$

6. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$. Be sure to clearly indicate the characteristic equation, and which eigenvalues and eigenvectors go together. (20 points)

$$(4-\lambda)(1-\lambda)+2 = \lambda^2-5\lambda+4+2 =$$

$$\boxed{\lambda^2-5\lambda+6=0}$$

$$(\lambda-2)(\lambda-3)=0 \qquad \lambda_1=2,\ \lambda_2=3$$

$\lambda_1=2$

$$\begin{bmatrix} 4-2 & -2 \\ 1 & 1-2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \qquad x_1 = x_2 \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{v_1}$$

$\lambda_2=3$

$$\begin{bmatrix} 4-3 & -2 \\ 1 & 1-3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \qquad x_1 = 2x_2 \qquad \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \vec{v_2}$$

7. Given that A and B are 9×9 matrices with det A = -3 and det B = 5, find the following. (5 points each)

a) det AB $\quad (-3)(5) = -15$

b) det A^{-1} $\quad = \frac{1}{-3}$

c) det $4B^T$ $\quad (4)^9 (5) = 1{,}310{,}720$

8. Given $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 3 \end{bmatrix}$, *and* $\mathbf{u}_2 = \begin{bmatrix} -4 \\ 5 \\ -3 \\ 2 \end{bmatrix}$ and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Determine if $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for *W*. If it is not, make it an orthogonal basis using the Gram-Schmidt process. (20 points)

$$\vec{u_1} \cdot \vec{u_2} = (1)(-4) + (-4)(5) + 0 + (3)(2) =$$
$$-4 - 20 + 6 = -18 \text{ not orthogonal}$$

$$\vec{u_1} = \vec{v_1}$$

$$\vec{v_2} = \begin{bmatrix} -4 \\ 5 \\ -3 \\ 2 \end{bmatrix} - \left(\frac{\begin{bmatrix} -4 \\ 5 \\ -3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -4 \\ 0 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ -4 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -4 \\ 0 \\ 3 \end{bmatrix}} \right) \begin{bmatrix} 1 \\ -4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \\ -3 \\ 2 \end{bmatrix} - \left(\frac{-\cancel{18}^{\,9}}{\cancel{26}_{\,13}} \right) \begin{bmatrix} 1 \\ -4 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -4 \\ 5 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 9/13 \\ -36/13 \\ 0 \\ 27/13 \end{bmatrix} = \begin{bmatrix} -43/13 \\ 29/13 \\ -3 \\ 53/13 \end{bmatrix}$$

$$\begin{bmatrix} -43 \\ 29 \\ -39 \\ 53 \end{bmatrix}$$

$$\vec{v_1} \cdot \vec{v_2} = (1)(-43) + (-4)(29) + (0)(-39) + 3(53) = 0$$

9. Given the basis of W in question #8, and the vector $\vec{y} = \begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ decompose this vector into $\vec{y} = \vec{y_{\parallel}} + \vec{y_{\perp}}$ with $\vec{y_{\parallel}} = proj_W \vec{y}$. (15 points)

You may use either the original basis or the orthogonal one. I'll use the original, but your answers should be the same.

$$\vec{y_{\parallel}} = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) u_2 = \left(\frac{5-8}{26}\right)\begin{bmatrix} 1 \\ -4 \\ 0 \\ 3 \end{bmatrix} + \left(\frac{-20+10-3+0}{16+25+9+4}\right)\begin{bmatrix} -4 \\ 5 \\ -3 \\ 2 \end{bmatrix} =$$

$$\left(\frac{-3}{26}\right)\begin{bmatrix} 1 \\ -4 \\ 0 \\ 3 \end{bmatrix} + \left(\frac{-13}{54}\right)\begin{bmatrix} -4 \\ 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 595/702 \\ -521/702 \\ 13/18 \\ -581/702 \end{bmatrix} = \vec{y_{\parallel}}$$

$$\vec{y_{\perp}} = \begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \vec{y_{\parallel}} = \begin{bmatrix} 2915/702 \\ 1925/702 \\ 5/18 \\ 581/702 \end{bmatrix}$$

10. Given the matrix $A = \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix}$ and $Q = \begin{bmatrix} -2/7 & 5/7 \\ 5/7 & 2/7 \\ 2/7 & -4/7 \\ 4/7 & 2/7 \end{bmatrix}$ containing an orthonormal basis for Col A, find a QR factorization of A. [Hint: Find R.] (10 points)

$$Q^T = \begin{bmatrix} -2/7 & 5/7 & 2/7 & 4/7 \\ 5/7 & 2/7 & -4/7 & 2/7 \end{bmatrix}$$

$$Q^T A = \begin{bmatrix} -2/7 & 5/7 & 2/7 & 4/7 \\ 5/7 & 2/7 & -4/7 & 2/7 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix} = R$$

11. Determine if each statement is True or False. (2 points each)

a. T F If vectors $\vec{v_1}, \vec{v_2}, \ldots \vec{v_p}$ span a subspace W and if $\vec{x}$ is orthogonal to each $\vec{v_j}$ for j=1...p, then $\vec{x}$ in $W^{\perp}$.

b. T F If $\vec{y}$ is in a subspace W, then the orthogonal projection of $\vec{y}$ onto W is $\vec{y}$ itself.

c. T F The pivot positions in a matrix depend on whether row interchanges take place. unique pivot positions

d. T F Matrix multiplication is commutative. $AB \neq BA$ generally

e. T F If the distance from $\vec{u}$ to $\vec{v}$ equals the distance from $\vec{u}$ to $-\vec{v}$ then $\vec{u}$ and $\vec{v}$ are orthogonal.

f. T F If a system of equations has a free variable then it has a unique solution. nonunique

g. T F If A is a $n \times n$ matrix, then A is invertible. a prerequisite but not a guarantee

h. T F If two vectors are orthogonal, they are linearly independent.

i. T F If matrix B is formed by multiplying matrix A by -1, then det B = -det A. only if matrix has odd dimensions

j. T F A linearly independent set in a subspace H is a basis for H. must also span

k. T F If A and B are row equivalent, then their column spaces are the same. row space yes, column space no.

l. T F The row space of A is the same as the column space of A^T.

m. T F An nxn matrix can have more than n eigenvalues. no more than n

n. T F The elementary row operations of A do not change its eigenvalues. does change

o. T F If the columns of A are linearly independent, then the equation $A\vec{x} = \vec{b}$ has exactly one least-squares solution.

p. T F A least-squares solution of $A\vec{x} = \vec{b}$ is the point in the column space of A closest to $\vec{b}$.

Name________________________________

Math 2568, Final Exam – Part 2, Spring 2013

Instructions: On this portion of the exam, you *may* use a calculator to perform elementary matrix operations. Support your answers with work (reproduce the reduced matrices from your calculator) or other justification for full credit.

1. Find a least squares solution for the set of points {(1,0), (2,2), (2,3), (3,4), (4,6), (5,9)} (12 points)

$y = \beta_0 + \beta_1 x + \beta_2 x^2$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 4 \\ 6 \\ 9 \end{bmatrix} \quad (A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} -1.4 \\ 1.617 \\ .083 \end{bmatrix} = \begin{bmatrix} -14/10 \\ 97/60 \\ 1/12 \end{bmatrix}$$

$$y = -14/10 + 97/60\, x + 1/12\, x^2$$

$$y = .083x^2 + 1.617x - 1.4$$

2. Let A = $\begin{bmatrix} 1 & 3 & 0 & 9 & -7 \\ -1 & -4 & 2 & 6 & 0 \\ 1 & 6 & 1 & -2 & 0 \\ 2 & 0 & 3 & 0 & 1 \\ 6 & 0 & -5 & -3 & -1 \\ 1 & 12 & 8 & 2 & 4 \end{bmatrix}$ $\text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

a. Determine if the columns of A form a linearly independent or dependent set and justify your answer. (4 points)

yes. 5 columns, 5 pivots

b. Determine if the columns of A span $\mathbb{R}^6$. Justify your answer. (4 points)

no, need 6 columns only 5

c. Use the information obtained in parts a and b to determine if the linear transformation $T: \vec{x} \in R^5 \mapsto A\vec{x} \in R^6$ is one-to-one or onto. Justify your answer. (4 points)

one-to-one, but not onto

3. Let us define an inner product on functions by $f \cdot g = \int_{-1}^{1} f(x)g(x)dx$. Show that the polynomials {1, x} form an orthogonal basis for P_1 using this inner product. Find a third polynomial, now in P_2, that will be orthogonal to the first two. (15 points)

$$\int_{-1}^{1} x(1)\,dx = \int_{-1}^{1} x\,dx = 0 \quad \underline{\text{odd}}$$

$$\int_{-1}^{1}(1)(a_0 + a_1x + a_2x^2)\,dx = \int_{-1}^{1} a_0 + \cancel{a_1x} + a_2x^2\,dx = a_0x + \frac{a_2}{3}x^3\Big|_{-1}^{1} =$$

$$a_0 + \frac{a_2}{3} + a_0 + \frac{a_2}{3} = 2a_0 + \frac{2a_2}{3} = 0$$

$$\int_{-1}^{1}(x)(a_0 + a_1x + a_2x^2)\,dx = \int_{-1}^{1} \cancel{a_0x} + a_1x^2 + \cancel{a_2x^3}\,dx$$

$$\int_{-1}^{1} a_1x^2\,dx = \frac{a_1}{3}x^3\Big|_{-1}^{1} = \frac{a_1}{3} + \frac{a_1}{3} = \frac{2a_1}{3} = 0 \Rightarrow a_1 = 0$$

$$2a_0 = -\frac{2a_2}{3} \Rightarrow a_0 = -\frac{a_2}{3} \qquad \text{let } a_2 = 3 \quad a_0 = 1$$

$$p(x) = 1 + 0x - 3x^2 = \boxed{1 - 3x^2}$$ also orthogonal to 1, x.

4. Given the basis {1, t, 1-3t²}, find the representation of p(t)=4t²+17t-25 in this basis. (10 points)

$$P_B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$[x]_B = P_B^{-1}[x] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}^{-1} \begin{bmatrix} -25 \\ 17 \\ 4 \end{bmatrix} = \begin{bmatrix} -71/3 \\ 17 \\ -4/3 \end{bmatrix}$$

5. Answer the following questions as fully as possible, and justify your answer.

a. If A is a 5x3 matrix with three pivot positions, does the equation $A\vec{x} = \vec{0}$ have a solution? If so, is it trivial or non-trivial? (5 points)

yes. it is trivial

one in each column ⇒ linearly independent

⇒ solution is unique

b. Determine if the set $H = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$ forms a basis for $\mathbb{R}^3$. Justify your answer. (7 points)

reduces to identity so it is independent

B spans space, yes is a basis

c. Prove that the space defined by $W = \left\{\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}, a, b, c \ real\right\}$ is or is not a vector space. (7 points)

it is if $a, b, c = 0$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the ~~zero~~ vector

closed under addition $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} + \begin{bmatrix} d & 0 \\ e & f \end{bmatrix} = \begin{bmatrix} a+d & 0 \\ b+e & c+f \end{bmatrix}$

$w/ (a+d), (b+e), (c+f)$ real

closed under scalar multiplication $k\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} = \begin{bmatrix} ka & 0 \\ kb & kc \end{bmatrix}$

$\Rightarrow w/ (ka), (kb), (kc)$ real.

d. For the stochastic matrix $P=\begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix}$, find the equilibrium vector by hand, and show by multiplication that the vector is the correct equilibrium vector. You may check your answer in the calculator, but I want to see work for full credit. (7 points)

$(P-I) = \begin{bmatrix} -.4 & .2 \\ .4 & -.2 \end{bmatrix}$ $\quad .4x_1 - .2x_2 = 0$

$\frac{.4x_1}{.4} = \frac{.2x_2}{.4} \Rightarrow x_1 = \frac{x_2}{2}$

$x_2 = x_2$

$q = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$ (must add to 1)

$Pq = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix}\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} \frac{6}{10}\cdot\frac{1}{3} + \frac{2}{10}\cdot\frac{2}{3} \\ \frac{4}{10}\cdot\frac{1}{3} + \frac{8}{10}\cdot\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1/5 + 2/15 \\ 2/15 + 8/15 \end{bmatrix} = \begin{bmatrix} 3/15 \\ 10/15 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

e. Explain why if U is a mxn matrix with orthonormal columns that the product U^TU is the mxm identity. [Hint: the solution is related to properties of dot products.] (10 points)

if columns of U given by $[u_1 \ u_2 \ u_3 \cdots]$

$U^TU = \begin{bmatrix} u_1\cdot u_1 & u_1\cdot u_2 & u_1\cdot u_3 & \cdots \\ u_2\cdot u_1 & u_2\cdot u_2 & u_2\cdot u_3 & \cdots \\ u_3\cdot u_1 & u_3\cdot u_2 & u_3\cdot u_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$

Since columns are orthogonal $u_i \cdot u_j = 0$ for $i \neq j$ and since columns are normal $u_i \cdot u_j = 1$ for $i = j$

6. Define the following terms as completely as possible. You may use examples in your explanations, but I do need more than just an example. (5 points each)
   a. What does it mean to be a *linear combination*?

a vector is a linear combination of other vectors if that vector can be obtained through at least one sum, scalar multiple or sum of scalar multiples of the other vectors.

   b. What does it mean for a linear transformation to be *onto* a space?

it means that every element in the range of the transformation can be obtained by at least one element in the domain.

   c. When we say that a system has a *trivial solution*, what do we mean?

we mean that the only solution is $\vec{0}$.

   d. What is the *rank* of a matrix?

the dimension of the row space.
alternatively, it is the number of pivot positions in the matrix.

e. What is meant by the term *orthogonal*? In R^n and in more general vector spaces?

orthogonal geometrically means perpendicular but more generally it means that given a specific inner product, two elements are orthogonal if their inner product is 0.