Linear Algebra Proof Set 1 Key.

1. let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ & $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ w/

u_i, v_i, w_i, c, d in $\mathbb{R}$

a) $\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}$ we can switch the order by the commutative property of real numbers, and thus

$= \begin{bmatrix} v_1+u_1 \\ v_2+u_2 \\ v_3+u_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \vec{v} + \vec{u}$ //

b) $(\vec{u} + \vec{v}) + \vec{w} = \begin{bmatrix} (u_1+v_1) \\ (u_2+v_2) \\ (u_2+v_3) \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} (u_1+v_1)+w_1 \\ (u_2+v_2)+w_2 \\ (u_3+v_3)+w_3 \end{bmatrix}$ we can move the parentheses by the associative property of real numbers, thus: =

$\begin{bmatrix} u_1+(v_1+w_1) \\ u_2+(v_2+w_2) \\ u_3+(v_3+w_3) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1+w_1 \\ v_2+w_2 \\ v_3+w_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right) = \vec{u} + (\vec{v} + \vec{w})$ //

c) $\vec{u} + \vec{0} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1+0 \\ u_2+0 \\ u_3+0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ by the additive identity for real numbers

and similarly $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0+u_1 \\ 0+u_2 \\ 0+u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \vec{0} + \vec{u}$ //

d) $\vec{u} + (-\vec{u}) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + (-1)\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} -u_1 \\ -u_2 \\ -u_3 \end{bmatrix} = \begin{bmatrix} u_1-u_1 \\ u_2-u_2 \\ u_3-u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$

$-\vec{u} + \vec{u}$ can be accounted for by the commutative property proved above in part a.

e) $c(\vec{u} + \vec{v}) = c\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) = c\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix} = \begin{bmatrix} c(u_1+v_1) \\ c(u_2+v_2) \\ c(u_3+v_3) \end{bmatrix}$ by the distributive property for real numbers this becomes $\begin{bmatrix} cu_1+cv_1 \\ cu_2+cv_2 \\ cu_3+cv_3 \end{bmatrix}$

$= \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} + \begin{bmatrix} cv_1 \\ cv_2 \\ cv_3 \end{bmatrix} = c\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + c\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = c\vec{u} + c\vec{v}$ //

f. $(c+d)\vec{u} = (c+d)\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} (c+d)u_1 \\ (c+d)u_2 \\ (c+d)u_3 \end{bmatrix}$ by the distributive property of real numbers we obtain

$$\begin{bmatrix} cu_1 + du_1 \\ cu_2 + du_2 \\ cu_3 + du_3 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} + \begin{bmatrix} du_1 \\ du_2 \\ du_3 \end{bmatrix} = c\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + d\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = c\vec{u} + d\vec{u} \;//$$

g. $c(d\vec{u}) = c\begin{bmatrix} du_1 \\ du_2 \\ du_3 \end{bmatrix} = \begin{bmatrix} cdu_1 \\ cdu_2 \\ cdu_3 \end{bmatrix} = (cd)\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = (cd)\vec{u}$

h. $1\vec{u} = 1\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1u_1 \\ 1u_2 \\ 1u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \vec{u} \;//.$

2. $f: \mathbb{R} \to \mathbb{R} \quad f(x) = mx + b$

a) $f(x+y) = m(x+y) + b = mx + my + b$

$f(x) + f(y) = mx + b + my + b = mx + my + 2b$

these are not equal unless $b = 0$.

b) $f(cx) = m(cx) + b = cmx + b$

$c f(x) = c(mx+b) = cmx + cb$

these are not equal unless $b=0$

c) $f(0) = m(0) + b = b$ which will only equal 0 as desired if $b=0$.

Thus $f(x) = mx$ is a linear transformation, but $f(x) = mx + b$ for $b \neq 0$ is not. //

3. let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$ $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$

and w/ $r, s, a_{ij}, b_{ij}, c_{ij} \in \mathbb{R}$

a) $A + B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{23}+b_{23} & a_{33}+b_{23} \end{bmatrix}$

by commutative property of real numbers we get

$$\begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} & b_{13}+a_{13} \\ b_{21}+a_{21} & b_{22}+a_{22} & b_{23}+a_{23} \\ b_{31}+a_{31} & b_{32}+a_{32} & b_{33}+a_{33} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = B + A \;//$$

2b. $(A+B)+C =$ (obtaining $A+B$ from part a)

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} (a_{11}+b_{11})+c_{11} & (a_{12}+b_{12})+c_{12} & (a_{13}+b_{13})+c_{13} \\ (a_{21}+b_{21})+c_{21} & (a_{22}+b_{22})+c_{22} & (a_{23}+b_{23})+c_{23} \\ (a_{31}+b_{31})+c_{31} & (a_{32}+b_{32})+c_{32} & (a_{33}+b_{33})+c_{33} \end{bmatrix}$$

using the associative property of real numbers on each entry we get

$$\begin{bmatrix} a_{11}+(b_{11}+c_{11}) & a_{12}+(b_{12}+c_{12}) & a_{13}+(b_{13}+c_{13}) \\ a_{21}+(b_{21}+c_{21}) & a_{22}+(b_{22}+c_{22}) & a_{23}+(b_{23}+c_{23}) \\ a_{31}+(b_{31}+c_{31}) & a_{32}+(b_{32}+c_{32}) & a_{33}+(b_{33}+c_{33}) \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} & b_{13}+c_{13} \\ b_{21}+c_{21} & b_{22}+c_{22} & b_{23}+c_{23} \\ b_{31}+c_{31} & b_{32}+c_{32} & b_{33}+c_{33} \end{bmatrix} = A + \left(\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \right)$$

$$= A + (B+C) \;//$$

c. $$A+O = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11}+0 & a_{12}+0 & a_{13}+0 \\ a_{21}+0 & a_{22}+0 & a_{23}+0 \\ a_{31}+0 & a_{32}+0 & a_{33}+0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= A \;//$$

d. $$r(A+B) = r \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix} = \begin{bmatrix} r(a_{11}+b_{11}) & r(a_{12}+b_{12}) & r(a_{13}+b_{13}) \\ r(a_{21}+b_{21}) & r(a_{22}+b_{22}) & r(a_{23}+b_{23}) \\ r(a_{31}+b_{31}) & r(a_{32}+b_{32}) & r(a_{33}+b_{33}) \end{bmatrix}$$

using the distributive property of real numbers on each entry

$$\begin{bmatrix} ra_{11}+rb_{11} & ra_{12}+rb_{12} & ra_{13}+rb_{13} \\ ra_{21}+rb_{21} & ra_{22}+rb_{22} & ra_{23}+rb_{23} \\ ra_{31}+rb_{31} & ra_{32}+rb_{32} & ra_{33}+rb_{33} \end{bmatrix} = \begin{bmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \\ ra_{31} & ra_{32} & ra_{33} \end{bmatrix} + \begin{bmatrix} rb_{11} & rb_{12} & rb_{13} \\ rb_{21} & rb_{22} & rb_{23} \\ rb_{31} & rb_{32} & rb_{33} \end{bmatrix}$$

$$= r \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + r \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = rA + rB \;//$$

e. $$(r+s)A = (r+s) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} (r+s)a_{11} & (r+s)a_{12} & (r+s)a_{13} \\ (r+s)a_{21} & (r+s)a_{22} & (r+s)a_{23} \\ (r+s)a_{31} & (r+s)a_{32} & (r+s)a_{33} \end{bmatrix} =$$

by distributive property of real numbers on each entry

$$\begin{bmatrix} ra_{11}+sa_{11} & ra_{12}+sa_{12} & ra_{13}+sa_{13} \\ ra_{21}+sa_{21} & ra_{22}+sa_{22} & ra_{23}+sa_{23} \\ ra_{31}+sa_{31} & ra_{32}+sa_{32} & ra_{33}+sa_{32} \end{bmatrix} =$$

2e cont'd.

$$= \begin{bmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \\ ra_{31} & ra_{32} & ra_{33} \end{bmatrix} + \begin{bmatrix} sa_{11} & sa_{12} & sa_{13} \\ sa_{21} & sa_{22} & sa_{23} \\ sa_{31} & sa_{32} & sa_{33} \end{bmatrix} = r \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + s \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

4

$$= rA + sA \text{ //}$$

f. $$r(sA) = r\left(s \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right) = r \begin{bmatrix} sa_{11} & sa_{12} & sa_{13} \\ sa_{21} & sa_{22} & sa_{23} \\ sa_{31} & sa_{32} & sa_{33} \end{bmatrix} = \begin{bmatrix} rsa_{11} & rsa_{12} & rsa_{13} \\ rsa_{21} & rsa_{22} & rsa_{23} \\ rsa_{31} & rsa_{32} & rsa_{33} \end{bmatrix}$$

$$= (rs) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = (rs)A \text{ //}$$

g. for the sake of space I'll check the multiplication associative property of just 2x2 matrices (as well as parts h, i, j, and later part o) for part g. only let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

$a_{ij}, b_{ij}, c_{ij} \in \mathbb{R}$

$$A(BC) = A\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} (b_{11}c_{11} + b_{12}c_{21}) & (b_{11}c_{12} + b_{12}c_{22}) \\ (b_{21}c_{11} + b_{22}c_{21}) & (b_{21}c_{12} + b_{22}c_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(b_{11}c_{11} + b_{12}c_{21}) + a_{12}(b_{21}c_{11} + b_{22}c_{21}) & a_{11}(b_{11}c_{12} + b_{12}c_{22}) + a_{12}(b_{21}c_{12} + b_{22}c_{22}) \\ a_{21}(b_{11}c_{11} + b_{12}c_{21}) + a_{22}(b_{21}c_{11} + b_{22}c_{21}) & a_{21}(b_{11}c_{12} + b_{12}c_{22}) + a_{22}(b_{21}c_{12} + b_{22}c_{22}) \end{bmatrix}$$

$$(AB)C = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right) C = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11}(a_{11}b_{11} + a_{12}b_{21}) + c_{21}(a_{11}b_{12} + a_{12}b_{22}) & c_{12}(a_{11}b_{11} + a_{12}b_{21}) + c_{22}(a_{11}b_{12} + a_{12}b_{22}) \\ c_{11}(a_{21}b_{11} + a_{22}b_{21}) + c_{21}(a_{21}b_{12} + a_{22}b_{22}) & c_{12}(a_{21}b_{11} + a_{22}b_{21}) + c_{22}(a_{21}b_{12} + a_{22}b_{22}) \end{bmatrix}$$

Careful comparison of the 4 products in each entry of $(AB)C$ and $A(BC)$ reveal that each entry is indeed identical after distribution. //

the 3x3 proof is the same, though the entries for the final matrix have nine products each and are longer to write out than the width of the page.

3h. $A(B+C) = A\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}(b_{11}+c_{11}) + a_{12}(b_{21}+c_{21}) & a_{11}(b_{12}+c_{12}) + a_{12}(b_{22}+c_{22}) \\ a_{21}(b_{11}+c_{11}) + a_{22}(b_{21}+c_{21}) & a_{21}(b_{12}+c_{12}) + a_{22}(b_{22}+c_{22}) \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} + \begin{bmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{11}c_{11} + a_{12}c_{21} & a_{11}b_{12} + a_{11}c_{12} + a_{12}c_{22} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} + a_{21}c_{11} + a_{22}c_{21} & a_{21}b_{12} + a_{22}b_{22} + a_{21}c_{12} + a_{22}c_{22} \end{bmatrix}$$

Careful inspection of the results from both process reveal sums of identical products in each entry and so the resulting matrices are equal. //

3i $(B+C)A = \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} =$

$$\begin{bmatrix} a_{11}(b_{11}+c_{11}) + a_{21}(b_{12}+c_{12}) & a_{12}(b_{11}+c_{11}) + a_{22}(b_{12}+c_{12}) \\ a_{11}(b_{21}+c_{21}) + a_{21}(b_{22}+c_{22}) & a_{12}(b_{21}+c_{21}) + a_{22}(b_{22}+c_{22}) \end{bmatrix}$$

$$BA + CA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} =$$

$$\begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix} + \begin{bmatrix} c_{11}a_{11} + c_{12}a_{21} & c_{11}a_{12} + c_{12}a_{22} \\ c_{21}a_{11} + c_{22}a_{21} & c_{21}a_{12} + c_{22}a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} + c_{11}a_{11} + c_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} + c_{11}a_{12} + c_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} + c_{21}a_{11} + c_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} + c_{21}a_{12} + c_{22}a_{22} \end{bmatrix}$$

Comparing the results of the endproducts here shows that the four products in each entry are indeed the same. //

3j. $r(AB) = r\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) = r\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}$

$= \begin{bmatrix} r(a_{11}b_{11}+a_{12}b_{21}) & r(a_{11}b_{12}+a_{12}b_{22}) \\ r(a_{21}b_{11}+a_{22}b_{21}) & r(a_{21}b_{12}+a_{22}b_{22}) \end{bmatrix} = \begin{bmatrix} ra_{11}b_{11}+ra_{12}b_{21} & ra_{11}b_{12}+ra_{12}b_{22} \\ ra_{21}b_{11}+ra_{22}b_{21} & ra_{21}b_{12}+ra_{22}b_{22} \end{bmatrix}$

$(rA)B = \left(r\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) = \begin{bmatrix} ra_{11} & ra_{12} \\ ra_{21} & ra_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} =$

$\begin{bmatrix} ra_{11}b_{11}+ra_{12}b_{21} & ra_{11}b_{12}+ra_{12}b_{22} \\ ra_{21}b_{11}+ra_{22}b_{21} & ra_{21}b_{12}+ra_{22}b_{22} \end{bmatrix}$

$A(rB) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\left(r\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} rb_{11} & rb_{12} \\ rb_{21} & rb_{22} \end{bmatrix} =$

$\begin{bmatrix} a_{11}rb_{11}+a_{12}rb_{21} & a_{11}rb_{12}+a_{12}rb_{22} \\ a_{21}rb_{11}+a_{22}rb_{21} & a_{21}rb_{12}+a_{22}rb_{22} \end{bmatrix}$

these results are indeed all equal. //

k. $I_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A$

$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A$ //

l. $(A^T)^T = \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T\right)^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A$ //

m. $(A+B)^T = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}^T = \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} & a_{31}+b_{31} \\ a_{12}+b_{12} & a_{22}+b_{22} & a_{32}+b_{32} \\ a_{13}+b_{13} & a_{23}+b_{23} & a_{33}+b_{33} \end{bmatrix}$

$= \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{bmatrix} = A^T + B^T$ //

3n. $(rA)^T = \begin{bmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \\ ra_{31} & ra_{32} & ra_{33} \end{bmatrix}^T = \begin{bmatrix} ra_{11} & ra_{21} & ra_{31} \\ ra_{12} & ra_{22} & ra_{32} \\ ra_{13} & ra_{23} & ra_{33} \end{bmatrix} =$ ⑦

$r\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = rA^T$ //

o. $(AB)^T = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right)^T = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}^T$

$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{21}b_{11} + a_{22}b_{21} \\ a_{11}b_{12} + a_{12}b_{22} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

$B^TA^T = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} =$

$\begin{bmatrix} b_{11}a_{11} + b_{21}a_{12} & b_{11}a_{21} + b_{21}a_{22} \\ b_{12}a_{11} + b_{22}a_{12} & b_{12}a_{21} + b_{22}a_{22} \end{bmatrix}$

these are equivalent matrices. //