Math 2568 Proof Set 4 Key

1. if λ is an eigenvalue of A then consider the matrix $A-\lambda I$. also consider $(A-\lambda I)^T = A^T - (\lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$ since $\det(A-\lambda I) = 0$ and $\det A^T = \det A$ this means that $\det(A^T - \lambda I) = 0$ so λ is also an eigenvalue of A^T.

2. If 0 is an eigenvalue of A then the equation $(A-\lambda I)x = 0$ reduces to $A\vec{x} = 0$. This means that A has a Nullspace w/ non-zero dimension and so it is not invertible. By contrast if 0 is not an eigenvalue of A then there is similarity transformation w/which the matrix is triangular and all entries on the diagonal are non-zero. Such a matrix can be reduced through row operations to the identity and so the matrix is invertible. By the same reasoning. The matrix will have a non-zero determinant in triangular form if the entries on the diagonal are non-zero, and since we know the similarity transformation preserves eigenvalues, we know that the matrix is invertible since a triangular matrix w/ no zeros on the diagonal can be row reduced to the identity If we know that a matrix is invertible, by contrast, then we know the equation $A\vec{x} = 0$ has only the trivial solution since the dimension of the nullspace is 0, the matrix cannot have 0 as an eigenvalue since that would imply the nullspace dimension was non-zero. If A is invertible, there is a basis under which A is diagonal and so the determinant will be the product of the diagonal entries. If A is invertible, these diagonal entries can be scaled to reduce to the identity, but this determinant is

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non-zero, and so det A must also be non-zero.

3. if A & B are similar then $A = PBP^{-1}$. taking the determinant we have $\det A = \det(PBP^{-1}) = \det P \det B \det P^{-1}$ $= \det P \cdot \det P^{-1} \det B = \det P \cdot \frac{1}{\det P} \cdot \det B = \det B.$

Therefore $\det A = \det B$. If A & B have the same determinant they must have the same characteristic equation, and so therefore have the same polynomial. if $A\vec{x} = \lambda\vec{x}$ then $PBP^{-1}\vec{x} = \lambda\vec{x}$,

$\Rightarrow \cancel{P^{-1}P}BP^{-1}\vec{x} = P^{-1}\lambda\vec{x} \Rightarrow B[P^{-1}\vec{x}] = \lambda[P^{-1}\vec{x}]$ $P^{-1}\vec{x}$ are the eigenvectors of B, call them $\hat{\vec{x}}$ and so we have $B\hat{\vec{x}} = \lambda\hat{\vec{x}}$

and so $\det(A-\lambda I)=0$ and $\det(B-\lambda I)=0$ have the same values of λ.

4. $\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix} \Rightarrow A-\lambda I = \begin{bmatrix} 1-\lambda & -6 \\ 2 & -6-\lambda \end{bmatrix}$ $\det(A-\lambda I) = (1-\lambda)(-6-\lambda)+12$

$= \lambda^2 + 5\lambda + 6 = 0 \Rightarrow (\lambda+3)(\lambda+2)=0 \Rightarrow \lambda=-2, \lambda=-3$

$\begin{bmatrix} 1+2 & -6 \\ 2 & -6+2 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 2 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ $x_1 - 2x_2 = 0 \Rightarrow$ $x_1 = 2x_2$, $x_2 = x_2$ $\vec{V}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1+3 & -6 \\ 2 & -6+3 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 2 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}$ $2x_1 - 3x_2 = 0 \Rightarrow$ $x_1 = \frac{3}{2}x_2$, $x_2 = x_2$ $\vec{V}_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

2 vectors ⇒ bases for $\mathbb{R}^2$ independent.

$P = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ $D = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$ $P^{-1} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$PDP^{-1} = \begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}$ and likewise $P^{-1}AP = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$

therefore A is diagonalizable as $\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$ using the similarity transformation given by $P = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

5. Suppose that A is both diagonalizable and invertible. - (3)
then it follows that $A = PDP^{-1}$. Since A is invertible then A^{-1} is defined and so is $(PDP^{-1})^{-1} = A^{-1} = (P^{-1})^{-1} D^{-1} P^{-1} = PD^{-1}P^{-1}$. Since $(A^{-1})^{-1} = A$ we know that A is invertible. and Since $A^{-1} = PD^{-1}P^{-1}$ is a similarity transformation we need only show that D^{-1} is diagonal. Consider the diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ \vdots & d_2 & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$ w/ entries $d_1, d_2, \ldots d_n$ on the diagonal

Since $DD^{-1} = I$ $\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ \vdots & d_2 & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \cdot D^{-1} = I$ thus $\det(DD^{-1})$

$= \det I \Rightarrow \det D \det D^{-1} = 1 \Rightarrow \det D^{-1} = \dfrac{1}{\det D}$.

we know that $\det D = d_1 \cdot d_2 \cdots d_n$ the product of the diagonal entries. and so $\operatorname{Det} D^{-1} = \frac{1}{d_1} \cdot \frac{1}{d_2} \cdots \frac{1}{d_n}$ the product of the reciprocals. Thus D^{-1} must be a diagonal matrix w/ the values $1/d_i$ on the diagonal Since the d_i were the eigenvalues of D and $1/d_i$ are the eigenvalues of D^{-1}!

6. $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ c a scalar in $\mathbb{R}$

a. $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$
$\vec{v} \cdot \vec{u} = v_1u_1 + v_2u_2 + v_3u_3$ but these are the same since multiplication is commutative in $\mathbb{R}$

b. $(\vec{u} + \vec{v}) \cdot \vec{w} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = (u_1+v_1)w_1 + (u_2+v_2)w_2 + (u_3+v_3)w_3$

$= u_1w_1 + v_1w_1 + u_2w_2 + v_2w_2 + u_3w_3 + v_3w_3$

$= (u_1w_1 + u_2w_2 + u_3w_3) + (v_1w_1 + v_2w_2 + v_3w_3)$

$= \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

6c. $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(c\vec{v})$

$$(c\vec{u})\cdot\vec{v} = \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = cu_1v_1 + cu_2v_2 + cu_3v_3$$

$$c(\vec{u}\cdot\vec{v}) = c(u_1v_1 + u_2v_2 + u_3v_3) = cu_1v_1 + cu_2v_2 + cu_3v_3$$

by distributive property

$$\vec{u}\cdot(c\vec{v}) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} cv_1 \\ cv_2 \\ cv_3 \end{bmatrix} = u_1cv_1 + u_2cv_2 + u_3cv_3 =$$

$cu_1v_1 + cu_2v_2 + cu_3v_3$ by commutativity on $\mathbb{R}$

d. $\vec{u}\cdot\vec{u} = u_1^2 + u_2^2 + u_3^2$ The sum of positive terms must be ≥ 0.

Similarly if $u_1^2 + u_2^2 + u_3^2 = 0$ then clearly u_1, u_2, u_3 must all $= 0$ so $\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$.

7. if y is orthogonal to both $\vec{u}$ and $\vec{v}$ then $\vec{y}\cdot\vec{u} = 0$ and $\vec{y}\cdot\vec{v} = 0$

But then since $\vec{y}(\vec{u}+\vec{v}) = \vec{y}\cdot\vec{u} + \vec{y}\cdot\vec{v} = 0 + 0 = 0$ so $\vec{y}$ is orthogonal to $\vec{u}+\vec{v}$ as well.

8. suppose that $\vec{x}$ is in both W and $W^{\perp}$. any vector in W is orthogonal to any vector in $W^{\perp}$. So $\vec{w}_1$ in W and $\vec{w}_2$ in $W^{\perp}$ have $\vec{w}_1\cdot\vec{w}_2 = 0$. But since $\vec{x}$ is in both W and $W^{\perp}$ then $\vec{x}\cdot\vec{x} = 0$ but this means that $\vec{x}$ must $= \vec{0}$

9. if $\{\vec{w}_1, \vec{w}_2 \ldots \vec{w}_p\}$ is an orthogonal basis for W then each vector w_i is orthogonal to w_j. If $\{\vec{v}_1, \ldots \vec{v}_q\}$ is an orthogonal basis for $W^{\perp}$ then each vector $\vec{v}_i$ is orthogonal to $\vec{v}_j$. By definition, all vectors in W are orthogonal to all vectors in $W^{\perp}$. Therefore $\{\vec{w}_1, \vec{w}_2 \ldots \vec{w}_p, \vec{v}_1, \vec{v}_2 \ldots \vec{v}_q\}$ is a set of orthogonal set in $\mathbb{R}^n$. It is a basis since

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all orthogonal vectors are linearly independent and since $W \cup W^\perp$ spans $\mathbb{R}^n$, the set of vectors that span both also span $\mathbb{R}^n$.

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