Math 1152, Exam #2, Spring 2014 Name KEY

Instructions: Show all work. Use exact answers unless specifically asked to round. You may check your answers in the calculator, but you must show work to get full credit. Incorrect answers with no work will receive no credit. Be sure to complete all the requested elements of each problem.

1. Verify that $y = c_1e^t + c_2te^t$ is a solution to the differential equation $y'' - 2y' + y = 0$. (8 points)

$$y' = c_1e^t + c_2e^t + c_2te^t$$

$$y'' = c_1e^t + c_2e^t + c_2e^t + c_2te^t = c_1e^t + 2c_2e^t + c_2te^t$$

$$y'' - 2y' + y = (c_1e^t + 2c_2e^t + c_2te^t) - 2(c_1e^t + c_2e^t + c_2te^t) + c_1e^t + c_2te^t = c_1e^t + 2c_2e^t + c_2te^t - 2c_1e^t - 2c_2e^t - 2c_2te^t + c_1e^t + c_2te^t =$$

$$e^t(c_1 - 2c_1 + c_1) + e^t(2c_2 - 2c_2) + te^t(c_2 - 2c_2 + c_2) = 0$$

2. Find the solution to the initial value problem $u'(x) = \frac{1}{x^2+16}$, $u(0) = 2$. (6 points)

$$\int du = \int \frac{dx}{x^2+16} \Rightarrow u = \frac{1}{4}\arctan\left(\frac{x}{4}\right) + C$$

$$u = 2 = \frac{1}{4}\arctan(0) + C \Rightarrow C = 2$$

$$u(x) = \frac{1}{4}\arctan\left(\frac{x}{4}\right) + 2$$

3. Suppose that the population of a city is increasing at 6% per year. It obeys the differential equation $\frac{dy}{dx} = 0.6y$. Solve the differential equation and write an equation that models the population of the city at time t if the initial population is 150,000. How many people will be living in the city 7years from now? (10 points)

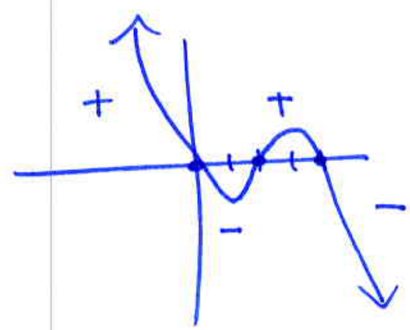
$$\int \frac{dy}{y} = \int .06\,dx \Rightarrow \ln y = .06t + C \Rightarrow y = y_0e^{.06t}$$

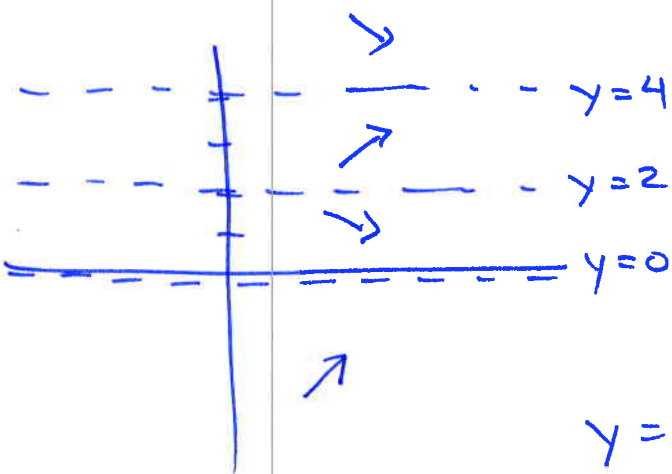
$$y(0) = 150{,}000 \Rightarrow y_0 = 150{,}000$$

$$150{,}000e^{.06(7)} = 228{,}294$$

4. Given the differential equation $\frac{dy}{dx} = y(4-y)(y-2)$. Sketch the phase plane for the equation and use that information to graph the key features of the direction field such as the equilibria (steady state solutions) and the sign of the slope in each region. Label each equilibrium as stable, unstable or semi-stable. (15 points)

$-(y-4)$





y=0 stable/attracting

y=2 unstable/ repelling (threshold)

y=4 stable/attracting (carrying capacity)

5. Solve the separable differential equation $y' = xe^{x-y}$. (7 points)

$$e^y \frac{dy}{dx} = xe^x e^{-y} \cdot e^y$$

$$\int e^y\,dy = \int xe^x\,dx$$

$$e^y = xe^x - e^x + C \quad \text{or} \quad y = \ln|xe^x - e^x + C|$$

6. Use Euler's method to approximate the first **three** steps of a ten step approximation to the solution at x=2 for $\frac{dy}{dx} = \frac{2x+1}{y-3}$, from the initial position $y(1) = 5$. (9 points)

$\frac{2-1}{10} = \frac{1}{10} = .1$

$m_1 = \frac{2(1)+1}{5-3} = \frac{3}{2} = 1.5$

$y(1.1) = 1.5(.1) + 5 = 5.15$

$m_2 = \frac{2(1.1)+1}{5.15-3} = \frac{3.2}{2.15} = 1.488... = \frac{64}{43}$

$y(1.2) = \frac{64}{43}(.1) + 5.15 = 5.2988... = \frac{4557}{860}$

$m_3 = \frac{2(1.2)+1}{\frac{4557}{860}-3} \quad \frac{3.4}{2.2988} = 1.479... = \frac{2924}{1977}$

$y(1.3) = \frac{2924}{1977}(.1) + \frac{4557}{860} = 5.44673... \approx 5.45$

7. Find a parameterization of the line from $(-1,-3)$ to $(6,-16)$. Specify the restriction on the parameter necessary to limit the graph to just the segment of the line between the points. [Hint: there is more than one but I'll give you three bonus if you can find a parameterization that always stays on the segment between these two points for any value of the parameter.] (6 points)

$\Delta x = 6-(-1) = 7$ $\quad x = 7t-1$ $\quad 0 \le t \le 1$

$\Delta y = -16-(-3) = -13$ $\quad y = -13t-3$

bonus:

$-1 \le \cos t \le 1$

set (x_0, y_0) = midpoint of segment

midpoint $= \left(\frac{-1+6}{2}, \frac{-3-16}{2}\right) = \left(\frac{5}{2}, -\frac{19}{2}\right)$

set $\Delta x = \frac{1}{2}$ previous value

Δy = ditto

$x = \frac{7}{2}\cos t + \frac{5}{2}$

$y = -\frac{13}{2}\cos t - \frac{19}{2}$

Check $\cos t = 1$, $t = 0$

$\frac{7}{2} + \frac{5}{2} = 6$

$-\frac{13}{2} - \frac{19}{2} = -16$

$t = \pi$ (other extreme when $\cos t = -1$)

$-\frac{7}{2} + \frac{5}{2} = -1$

$+\frac{13}{2} - \frac{19}{2} = -3$

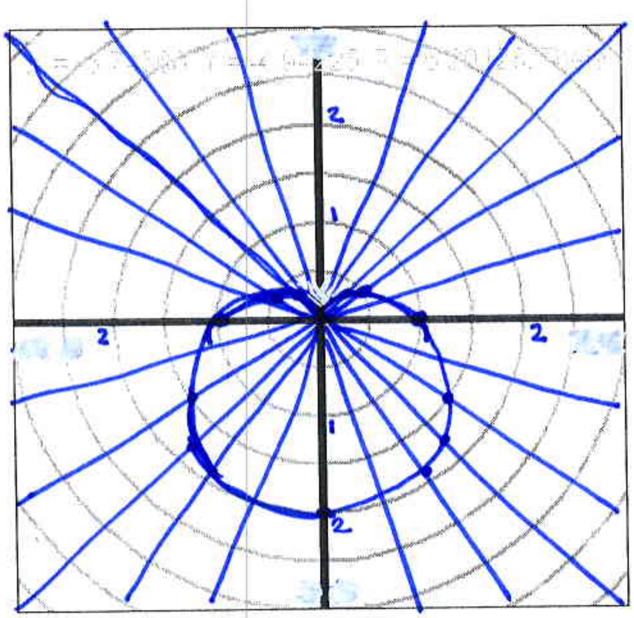
or $x = 7\sin^2 t - 1$

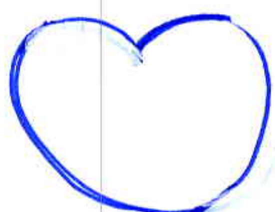
$y = -13\sin^2 t - 3$

8. Sketch the graph of $r = 1 - \sin\theta$ on the graph below. You must plot at least 8 points with exact values. (8 points)

I let each ring $= r = .5$

θ	r
0	1
$\pi/6$	$1/2$
$\pi/4$	$1-\frac{\sqrt{2}}{2} \approx .292\ldots$
$\pi/3$	$1-\frac{\sqrt{3}}{2} \approx .133\ldots$
$\pi/2$	0
$2\pi/3$	$1-\frac{\sqrt{3}}{2}$
$3\pi/4$	$1-\frac{\sqrt{2}}{2}$
$5\pi/6$	$1/2$
π	1
$-\pi/6$	1.5
$-\pi/4$	$1+\frac{\sqrt{2}}{2} \approx 1.707$
$-\pi/3$	$1+\frac{\sqrt{3}}{2} \approx 1.866$
$-\pi/2$	2





9. Find the slope of the tangent line to the graph $r = 2 - 3\cos\theta$ when $\theta = \frac{\pi}{3}$. You may use the formula $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$. [Hint: to write the line, you may do it in rectangular coordinates, but you will have to convert the point on the polar graph to rectangular as well.] (10 points)

$r' = 3\sin\theta$

$$\frac{dy}{dx} = \frac{3\sin\theta \cdot \sin\theta + (2-3\cos\theta)\cos\theta}{3\sin\theta\cos\theta - (2-3\cos\theta)\sin\theta}$$

$\sin\pi/3 = \frac{\sqrt{3}}{2}$

$\cos\pi/3 = 1/2$

2, $\sqrt{3}$, $\pi/3$, 1

$$= \frac{3\left(\frac{\sqrt{3}}{2}\right)^2 + (2-3/2)(1/2)}{3\left(\frac{\sqrt{3}}{2}\right)(1/2) - (2-3/2)(\sqrt{3}/2)} =$$

$2-3\cos(\theta) =$ ($\theta = \pi/3$)

$2-3(1/2) = 1/2$

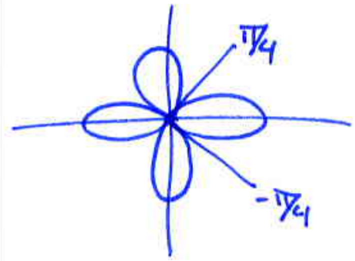
$(1/2, \pi/3) \Rightarrow \left(\frac{1}{2}(1/2), \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)$

$\Rightarrow \left(1/4, \frac{\sqrt{3}}{4}\right)$

$$\frac{\frac{9}{4}+\frac{1}{4}}{\frac{3\sqrt{3}}{4} - \frac{\sqrt{3}}{4}} = \frac{\frac{10}{4}}{\frac{2\sqrt{3}}{4}} = \frac{\cancel{10}^{5}}{\cancel{4}}\cdot\frac{\cancel{4}}{\cancel{2}\sqrt{3}} = \frac{5}{\sqrt{3}}$$

$$\boxed{y - \frac{\sqrt{3}}{4} = \frac{5}{\sqrt{3}}\left(x - 1/4\right)}$$

10. Find the area of one petal of the graph $r = 2\cos 2\theta$. (7 points)



$$A = 2\cdot\frac{1}{2}\int_0^{\pi/4}[2\cos 2\theta]^2\,d\theta =$$

$$= \int_0^{\pi/4} 4\cos^2 2\theta\,d\theta =$$

$$2\int_0^{\pi/4} 1+\cos 4\theta\,d\theta = 2\left[\theta+\frac{1}{4}\sin 4\theta\right]_0^{\pi/4}$$

$$= 2[\pi/4] = \boxed{\pi/2}$$

$0 = 2\cos 2\theta$

$\cos 2\theta = 0$

$2\theta = \pi/2, -\pi/2$, etc

$\theta = \pi/4, -\pi/4$, etc.

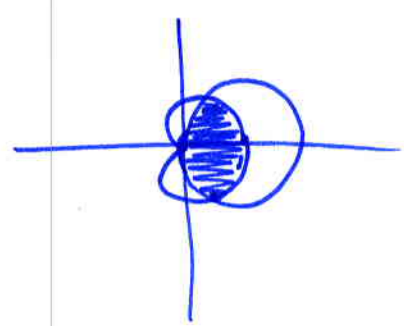
11. Find the area of the region common to the circle $r = 3\cos\theta$, and $r = 1+\cos\theta$. Sketch the region. (8 points)

$3\cos\theta = 1+\cos\theta$

$2\cos\theta = 1$

$\cos\theta = 1/2$

$\theta = \pi/3, -\pi/3$



$$\frac{2}{2}\int_0^{\pi/3}[1+\cos\theta]^2\,d\theta + \frac{2}{2}\int_{\pi/3}^{\pi/2}[3\cos\theta]^2\,d\theta =$$

$$\int_0^{\pi/3} 1+2\cos\theta+\underbrace{\cos^2\theta}_{1/2+\frac{1}{2}\cos 2\theta}\,d\theta + \int_{\pi/3}^{\pi/2} 9\underbrace{\cos^2\theta}_{1/2+\frac{1}{2}\cos 2\theta}\,d\theta =$$

$$\int_0^{\pi/3} 3/2+2\cos\theta+\frac{1}{2}\cos 2\theta\,d\theta + \int_{\pi/3}^{\pi/2}\frac{9}{2}+\frac{9}{2}\cos 2\theta\,d\theta =$$

$$\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\Big|_0^{\pi/3} + \frac{9}{2}\theta+\frac{9}{4}\sin 2\theta\Big|_{\pi/3}^{\pi/2} =$$

$$\frac{3}{2}(\pi/3)+\sqrt{3}+\frac{1}{4}\left(-\frac{\sqrt{3}}{2}\right)+\frac{9}{2}(\pi/6)-\frac{9}{4}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{5\pi/4+2\sqrt{3}}$$

12. Determine whether the polar conics below are circles, ellipses, parabolas or hyperbolas. What is the eccentricity of each graph. (3 points each)

a. $r = \frac{4}{1+\cos\theta}$ parabola

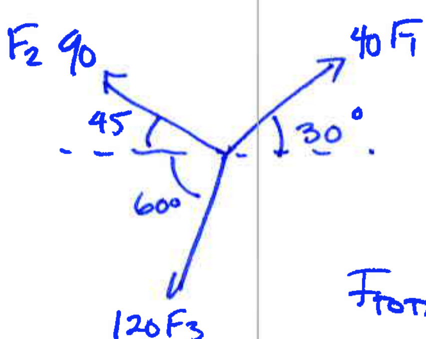
$e = 1$

b. $r = 2\sin\theta$ circle $e = 0$

c. $r = \frac{6}{3+2\sin\theta} = \frac{2}{1+\frac{2}{3}\sin\theta}$ ellipse $e = \frac{2}{3}$

d. $r = \frac{1}{1+2\cos\theta}$ hyperbola $e = 2$

13. Find the resulting force if three forces are applied to an object: F_1 has magnitude 40 lbs. and pulls in the direction 30° from above the positive horizontal axis, F_2 has magnitude 90 lbs. and pulls at an angle of 45° above the negative horizontal axis, and F_3 has a magnitude of 120 lbs. and pulls with an angle of 60° below the negative horizontal axis. You may round your answer to 2 decimal places. (10 points)



F_1 $\langle 40\cos 30°, 40\sin 30°\rangle = \langle 20\sqrt{3}, 20\rangle$

F_2 $\langle 90\cos 135°, 90\sin 135°\rangle = \langle -45\sqrt{2}, 45\sqrt{2}\rangle$

F_3 $\langle 120\cos 240°, 120\sin 240°\rangle = \langle -60, -60\sqrt{3}\rangle$

$\langle 20\sqrt{3} - 45\sqrt{2} - 60, 20 + 45\sqrt{2} - 60\sqrt{3}\rangle$

$F_{TOTAL} \approx \langle -88.99859416, -20.28343815\rangle$

$\|F_{TOTAL}\| \approx 91.28$

$\theta = \tan^{-1}\left(\frac{-20.28...}{-88.99...}\right) + 180° = 192.84°$ from $+x$ axis

or 12.84° below neg x-axis

14. Find the magnitude of the vector between the points (-2,1,0) and (3,9,11). (4 points)

$\vec{v} = \langle 3-(-2), 9-1, 11-0 \rangle = \langle 5, 8, 11 \rangle$

$\|\vec{v}\| = \sqrt{25+64+121} = \sqrt{210}$

15. Find the angle between the vectors $\langle -10,0,4 \rangle$ and $\langle -9,5,1 \rangle$. State your answers in radians to 4 decimal places. (5 points)

$\cos\theta = \dfrac{90+0+4}{\sqrt{116}\sqrt{107}} = \dfrac{94}{\sqrt{116 \cdot 107}}$ $\quad \theta \approx .56589$ radians

$\boxed{.5666}$

16. Find the magnitude of the work done by moving an object from (0,0,0) to (2,4,6) using a force of $\langle 10,4,3 \rangle$. (4 points)

$\vec{d} = \langle 2,4,6 \rangle$

$\vec{F} \cdot \vec{d} = 10 \cdot 2 + 4 \cdot 4 + 3 \cdot 6 = 20 + 16 + 18 = \boxed{54}$

17. Find the cross product of the vectors $2\hat{\imath} + 3\hat{\jmath} - 9\hat{k}$ and $\hat{\imath} + 2\hat{\jmath} - \hat{k}$. (6 points)

$\vec{u}$ $\vec{v}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -9 \\ 1 & 2 & -1 \end{vmatrix} = (-3+18)\hat{\imath} - (-2+9)\hat{\jmath} + (4-3)\hat{k}$$

$$= 15\hat{\imath} - 7\hat{\jmath} + \hat{k}$$

$$\langle 15, -7, 1 \rangle$$

18. Find the expression for the arc length of the exponential spiral given by $r = e^{\theta}$ between $[0,2\pi]$. Evaluate the expression in your calculator and give the solution with 4 decimal places. (8 points)

$$r' = e^{\theta}$$

$$L = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta =$$

$$= \int_0^{2\pi} \sqrt{[e^{\theta}]^2 + [e^{\theta}]^2}\, d\theta = \int_0^{2\pi} \sqrt{2e^{2\theta}}\, d\theta =$$

$$\sqrt{2} \int_0^{2\pi} e^{\theta}\, d\theta = \sqrt{2}\, e^{\theta} \Big|_0^{2\pi} = \sqrt{2}\left[e^{2\pi} - e^{0}\right] =$$

$$\sqrt{2}\left[e^{2\pi} - 1\right] \approx$$

$$755.8853\ldots$$